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Modeling The Components Of An Economy As A Complex Adaptive System

Richard J. Mickelsen

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THESIS

Richard J. Mickelsen, Captain, USAF

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AIR FORCE INSTITUTE OF TECHNOLOGY

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Richard J. Mickelsen, BS

Captain, USAF

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As A Complex Adaptive System

Richard J. Mickelsen, BS

Captain, USAF

Committee Membership:

Dr. Darryl K. Ahner
Chair

Dr. Richard F. Deckro
Member

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Abstract

Complex systems science is a relatively new discipline and has not been widely applied to the field of economics. Much of current economic theory relies on principles of constrained optimization and often fails to see economic variables as part of an interconnected network. While tools for forecasting economic indicators are based primarily on autoregressive techniques, these techniques are not always well-suited to predicting the future performance of highly volatile data sets such as the stock market. This research portrays the stock market as one component of a networked system of economic variables, with the federal funds rate acting as an exogenous influencing factor. Together these components form a complex adaptive system having nonlinear dynamics. The network is modeled using a system of differential equations, which are based on an expanded form of the logistic differential equation for populations with carrying capacities. An inverse problem is solved using the method of least squares, and the resulting coefficients are examined to determine the strength of relationships between the network components. The fitted model is then evaluated for adequacy and Euler's Forward Method is employed to predict the long-run behavior of the network. With this as a baseline, the research investigates several hypothetical scenarios to determine how the system reacts to changes in interest rates. Contributions and implications of the model are addressed in the context of U.S. national defense.

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To my wife, whose support and confidence mean everything to me.

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Richard J. Mickelsen

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MODELING THE COMPONENTS OF AN ECONOMY AS A COMPLEX ADAPTIVE SYSTEM

I. Introduction

1.1 Background

The stock market has been modeled in multitudinous ways for at least a century. Because of the great fortunes that have been made and lost in the market, millions of analysts spend countless hours trying to determine whether the market will go up or down in response to certain observable stimuli. Despite these efforts, however, the reasons behind many stock market movements remain elusive, and great bubbles and crashes still occur.

Perhaps the most notable market crash occurred in 1929. After stock prices surged during the "roaring twenties," the market dropped precipitously on 24 October 1929, losing 23% of its value in two days, initiating the beginning of the Great Depression. The market did not reach similar price levels again until the 1950's. During another great crash, that of 1987, stocks lost 22.61% of its value in one day. It was similarly devastating in the suddenness of its onset, even if the long-term economic effects were not as severe (Browning, 2007). Likewise, the "dot-com" bubble of the late 1990's ended in significant price drops, and ushered in the relatively mild U.S. recession of 2001. None of these events were entirely predicted, albeit many cautious individuals noticed the telltale signs that equities were overpriced in relation to their fundamental values and ready for a large correction.

While investor sentiment is often a major factor in dramatic stock market movements, the market itself tends to mirror many other economic phenomena since investors, in search for any clues that may reveal sources of profit, view these phenomena in action, or even anticipate them, and invest their money accordingly. A poignant example is the subprime mortgage crisis and housing market crash of 2007. As a result of decreased lending standards and low interest rates over 15 years, millions of Americans were able to buy homes that previously would have been inaccessible to them. As demand for real estate increased, prices began to climb and speculation among home buyers led to the creation of a housing bubble (Beachy, 2012: 10-13). The rapid gains in the housing sector yielded similar results for the stock market as companies in the banking, construction, and retail industries reaped the benefits of soaring home prices. Yet, the supply of new homes eventually outpaced demand and prices began to fall. When this occurred, combined with the maturing of risky short-term loans, such as those with balloon payments or variable interest rates, foreclosures skyrocketed as homeowners could no longer make payments on their expensive mortgages, and they could not sell the homes for a high enough price to satisfy their mortgage debts. The increasing foreclosures then affected banks, causing some to fail or carry bad debt. Falling home prices and home purchases heavily impacted the construction and retail industries. The crash cascaded through the economy, destroying business earnings, which resulted in lower anticipated stock returns. As a result, the stock market also crashed, reaching a low point in early 2009 and caused investors to lose approximately $16 trillion of value in less than a year ("The Great Recession," 2015).

The effects of the housing market crash of 2007 were felt throughout the American and global economies. As consumer and business credit tightened, corporate profits dropped, the economy stopped growing; the nation went into a recession that lasted nearly two years ("The Great Recession of 2008-09: Year in Review 2009," 2015). The federal government and Federal Reserve took extreme measures to ensure the economy did not fall into a depression. Congress enacted several spending plans designed to save failing businesses and inject money into the economy. In addition, the Federal Reserve dropped its interest rate from 5.25% in 2007 to zero by the end of 2008. Then, realizing the economy was in a recession and fearing that conditions would continue to worsen, the Federal Reserve began aggressively purchasing government bonds as part of its quantitative easing program. As a result, an artificial demand was created for treasury notes, forcing interest rates and bond yields to remain low while money flooded into the economy (Fisher, Richard W., 2010). Interest rates were kept near zero for one of the longest periods in U.S. history, without being raised again until December 2015.

In hindsight it is apparent the Fed's actions helped restore growth to the American economy. Unemployment gradually declined, housing prices began increasing, and corporate profits rebounded. The stock market also recovered. Aided by increasing business earnings, recovering consumer credit, and low yields in other asset classes, the S&P 500 climbed steadily from its low point in early 2009 until it reached an all-time high in 2015 – a climb that corresponded closely to the low interest rates and the mounting government debt held by the Federal Reserve as a result of its quantitative easing.

1.2 Problem Statement

More than anything, the U.S. housing crash illustrates the interconnected nature of the U.S. economy. Weaknesses in one area were closely tied to weaknesses in others. Likewise, imbalances in one segment of the economy caused significant imbalances to occur in others, and the ensuing correction back to underlying asset values was widespread. These observations suggest a clear, causal, though not necessarily direct, relationship between many components of the U.S. economy. Abundant models of different forms have already shown these relationships, but the exact combinations of variables that contribute to different economic situations, and the weight of influence they have on one another, can be difficult to identify and are often transitory in nature. Many of these models employ some form of autoregression to understand economic occurrences, but these rely on historical data to explain or predict the behavior of economic components, while neglecting the feedback mechanisms that are present in the larger system. Similarly, many prevailing theories view economic behavior at a single point in time and from a standpoint of constrained optimization. This viewpoint depends on many simplifying assumptions and disregards the interrelated structure of the economy that could be better modeled as a network that adapts and evolves over time (Foster, 2004b:4).

This research explores an alternative approach and examines economic behavior, and specifically stock market behavior, as a complex adaptive system. Traditionally reserved to the natural sciences, little published work has been done to depict an economy or its components as a networked, complex system, even though prominent economists such as Keynes and Schumpeter seem to have intuitively understood it as such (Foster, 2004b:24-27). While systems dynamics models do exist, they are distinct from the purely mathematical model of interest here and did not contribute to this research.

1.3 Research Objective and Implications

Importantly, this research shows that an economic system can be modeled as a complex adaptive system. In doing so, it conjectures a model in which the stock market is one component of a fully connected network and uses a system of differential equations to simulate the dynamics of the network components over time. The model and system of differential equations also include an exogenous variable to show how the effect of the Federal Reserve's monetary policy (i.e., the federal funds rate) influences the other components.

Because the relationships between the model components are unknown, the system of differential equations is first used to solve the inverse problem to find the coefficients that most accurately portray actual system behavior. Once these are identified, analysis is used to validate the model and test it in a series of hypothetical scenarios. Successful validation indicates that the model and its parameters can be used to simulate economic behavior and test the implementation of different monetary policies. More generally, it shows that a system of differential equations can be used on historical economic data to solve an inverse problem, fit a model with a logistic "carrying capacity" to that data, and show that economic behavior can be modeled as a complex adaptive system. Furthermore, it expands the variety of research options available to economists and national policymakers.

A model that accurately replicates the behavior of the stock market and other model components serves as evidence that financial markets behave as complex adaptive systems and that they can be modeled as such. The model is also useful as a starting point for further refinements that increase its accuracy and usefulness. Accordingly, significant insights are gained regarding the component relationships and the influence that interest rates have on them – information that would undoubtedly be of worth to the Federal Reserve, government policymakers, and private investors.

Additionally, these insights are useful in defense planning. Currently, the Treasury Department leads efforts to protect financial markets from internal and external threats, be they terrorist-related, illicit trading and money laundering, or malicious attacks on financial systems (U.S. Department of the Treasury, 2015). But increasingly, warfare is multidimensional, and a confrontation with another nation or non-state actor may include attacks on U.S. financial interests. The rise in cyber-attacks on U.S. infrastructure over the last decade testifies to this possibility (Yadron, 2015). In the event of such an attack, an increased knowledge of stock market dynamics, its reaction to external threats, and possible stabilization efforts would be beneficial.

Finally, from a purely scientific perspective, a successful model expands the field of complex systems science and demonstrates that some of the same principles that govern the natural world also govern the economic sphere.

1.4 Summary

Economists have struggled to understand how markets work for centuries. The most dominant theories have focused on the individual as a rational economic agent, and subsequent theories have built on this framework with reasonably successful results. Yet, the world has become more connected, diverse, and complicated, and it is apparent that large markets function as complex systems, capable of adapting to unique circumstances as they arise. Modeling economic systems as connected, adaptable networks may provide great insight into how the various segments of the economy function with one another.

The following chapters detail the methodology and analysis used to develop such a model. Chapter II provides an overview of economic warfare and the implications this research may have on national security toward identifying and resolving emerging threats. Chapter III presents literature relevant to the topic of the stock market, inverse problems, the method of least squares, and complex adaptive systems. The methodology used to develop and solve the model is discussed in Chapter IV, including the collection and processing of data, the structure of the model and its system of differential equations, and how the inverse problem is solved using the method of least squares. Chapter V discusses the results of the model, its validation, and several hypothetical scenarios. Lastly, Chapter VI summarizes the conclusions gleaned from the model and recommends topics for future research.

II. Economic Warfare

2.1 Chapter Overview

This chapter surveys the topic of economic and financial warfare as defined by U.S. military publications and a potential near-peer adversary. It illustrates the topic with several examples of offensive financial tactics undertaken by individual actors in the past that weakened entire countries.

2.2 Unrestricted Warfare

> Now that Asians have experienced the financial crisis in Southeast Asia, no one could be more affected by "financial war" than they have been. …After just one round of fighting, the economies of a number of countries had fallen back ten years. What is more, such a defeat on the economic front precipitates a near collapse of the social and political order. The casualties resulting from the constant chaos are no less than those resulting from a regional war, and the injury done to the living social organism even exceeds the injury inflicted by a regional war. …Thus, financial war is a form of non-military warfare which is just as terribly destructive as a bloody war, but in which no blood is actually shed. …[B]efore long, "financial warfare" will undoubtedly be an entry in the various types of dictionaries of official military jargon. Moreover, when people revise the history books on twentieth-century warfare in the early 21st century, the section on financial warfare will command the reader's utmost attention (Qiao & Wang, 1999: 51-52).

In 1999, two senior colonels of the People's Liberation Army of China wrote a book entitled *Unrestricted Warfare*. The main message of the book is that a less technologically advanced nation, such as China, can defeat a larger nation by creatively employing unconventional tactics. Unable to face the United States on the battlefield, they advocate that China instead gain advantages through financial warfare, in addition to proactively using U.S. and international law to its benefit; using terrorist attacks to destabilize its opponents' domestic security and political standing; or even employing detrimental environmental tactics to weaken its adversary. Through financial warfare they claim entire nations can be severely weakened. More than anything, they claim that a nation's perspective on war should encompass all relevant domains. Rather than having a dedicated military strategy that stands apart from political, economic, social, and environmental policies, they propose a composite, coordinated strategy that furthers the nation's struggle against other world interests on all fronts.

This perspective from the Chinese stands in contrast to America's approach to war. For many years the United States has recognized the need for a broad national strategy to combat international threats. For example, the current National Security Strategy (NSS) covers many topics, including military readiness, diplomatic engagement, and strategic communication. However, this strategy does not contain the same comprehensive focus that the Chinese senior colonels seem to advocate. The NSS addresses economic issues from a commerce perspective that concentrates on national economic growth, stability, and national competitiveness. It also discusses the need to assist in the economic development of other countries in order to provide those nations with stable societies that are not susceptible to terrorist or non-democratic pressures. Beyond that, financial markets are noted, but merely in the context of avoiding harmful "boom-bust" cycles, or preventing illicit trading and money laundering, especially by criminals and terrorists (Obama, 2010). It makes little mention of threats to American financial systems from foreign nations and says nothing in regards to financial warfare as an offensive strategy.

The U.S. government is interested in the economic security of the country, even if the efforts expended are limited. For example, the Treasury Department's Office of Intelligence and Analysis is the principal agency for gathering intelligence on foreign threats to the U.S. financial system, and for directing actions against those threats and other international targets identified by the federal government (Office of Intelligence and Analysis, 2014: 2-5). The intelligence they collect is disseminated to national leaders, including the military, and they are part of the national intelligence community. Additionally, the Secretary of the Treasury sits on the National Security Council (Obama, 2009). However, there is little overlap between national economic planning and defense planning from an operational standpoint.

Although the secondary and tertiary effects of U.S. military operations are analyzed by some offices within the Department of Defense – to include the economic effects – the Defense Department has little impact on economic issues either at home or internationally. Rather than use debilitating and targeted financial attacks on an adversary's markets in parallel with, or in place of, military action, as part of a coordinated strategy, military operations are more often the recourse after broad economic sanctions have failed to accomplish the desired objectives. Once armed intervention has already begun, military means are used in a more focused manner if it is determined that strategic or operational objectives can be met, either directly or indirectly, by targeting the economic generators of an adversary. In this context, U.S. joint doctrine addresses the economic aspect of warfare as both an operational variable of the battlespace, and, more broadly, as an instrument of national power. These applications are limited however, and the National Defense Strategy and National Military Strategy are silent on economic issues.

The next section reviews U.S. joint doctrine in regards to the economy as an operational variable and economic actions as an instrument of national power.

2.3 Operational Variables and the Instruments of National Power

U.S. joint doctrine states that military plans should consider American economic actions against other countries (Joint Chiefs of Staff, 2013: I10). Although offensive economic actions, i.e., sanctions, are normally initiated and implemented by the Treasury Department, these actions have a direct impact on the need for military force, the implementation of military force, and the results of such force. When applied against the United States, the economic actions of other countries may have an effect on the American warfighting ability, morale, or national will. Thus, it is expedient that military leaders and planners understand the dynamics of financial markets and the impact that outside influences can have on them. This section reviews U.S. joint doctrine as it relates to economics.

2.3.1 Instruments of National Power

The United States advances its foreign interests by employing the instruments of national power in pursuit of national strategic objectives. These instruments are defined by U.S. joint doctrine to be diplomatic, informational, military, and economic (DIME). The first, diplomatic, is described as the principal instrument for advancing national interests and influencing foreign nations and is managed by the U.S. State Department. The second, informational, includes all forms of media, including social media, that communicates an image or intent of the U.S. This content is strategically formulated and disseminated in an effort to influence key audience perceptions. The third instrument, military, is the coercive and deterrent arm of national power that serves to force adversaries to comply with U.S. desires. The last instrument, economic, is described as "the fundamental engine of general welfare, the enabler of a strong national defense" (Joint Chiefs of Staff, 2013: I6-I16). Thus, the Department of the Treasury is the lead for this aspect of national power internationally. It works with other U.S. agencies to advance American interests, and in today's globalized economy the effects of economic policies have wide ranging effects.

While separately implemented by independent government agencies, the four instruments of national power are complementary and are best utilized in an integrated approach. Joint Publication 1 states that

> The routine interaction of the instruments of national power is fundamental to US activities in the strategic security environment. … The (US government's) ability to achieve its national strategic objectives depends on employing the instruments of national power … in effective combinations and [in] all possible situations from peace to war. …To accomplish this integration, the Armed Forces interact with the other departments and agencies to develop a mutual understanding of the capabilities, limitations, and consequences of military and civilian actions. They also identify the ways in which military and nonmilitary capabilities best complement each other (Joint Chiefs of Staff, 2013: I13).

The joint doctrine continues, emphasizing that

> Political and military leaders must consider the employment of military force in operations characterized by a complex, interconnected, and global operational environment that affect the employment of capabilities and bear on the decisions of the commander. The addition of military force to coerce an adversary should be carefully integrated with the other instruments of national power to achieve our objectives (Joint Chiefs of Staff, 2013: I14).

The interconnected nature of the global environment therefore requires that U.S. leaders understand the potential side effects of the four instruments of power. Just as military action may affect the viability of the other three instruments, so also may economic policies affect or make future actions by the military necessary.

2.3.2 Operational Variables

The effect of the instruments of national power on a region of interest can be determined by measuring certain operational variables that describe the human environment within that region. The U.S. Army's *Field Manual 3-0, Operations (FM 3-0)*, describes these as political, military, economic, social, information, and infrastructure (PMESII). These variables are intentionally broad, as they seek to explain all aspects of the operational environment and how they relate to military campaigns and major operations (Headquarters Department Of The Army, 2008: 1.5-1.9). FM 3-0 gives general explanations for each of these variables, but of particular interest here is the economic variable, it's interdependency on the political variable, and how they both serve to shape an operational environment in major ways that may influence future U.S. military operations.

FM 3-0 describes the economic variable in terms of individual and group behaviors related to producing, distributing, and consuming resources. Specifically, these activities include trade, development (including foreign aid), banking and finance, monetary policy, and legal constraints on business activities. The economic variable is strongly connected to the political variable and can serve as an incentive for political action by a group of people if they perceive the opportunity exists. Aspects of the economic variable that may incentivize or disincentivize such action include technical knowledge, decentralized capital flows, investment, price fluctuations, debt, financial instruments, protection of property rights, black markets and underground economies (Headquarters Department Of The Army, 2008: 1.7).

In most countries, the activities mentioned above can be measured using widely available economic data and indices, such as gross domestic product (GDP) or gross national product; employment/unemployment levels; central bank policies that affect debt levels, interest rates, inflation, and currency valuations; equity markets; and foreign investment, to name a few. Most of this information can be accessed via the Federal Reserve databases, the International Monetary Fund, the World Bank, the CIA World Factbook, and others. More subjective measures of economic freedom, to include the protection of property rights (rule of law), the ease of starting a business (regulatory efficiency), and trade freedom (open markets) are collected by organizations such as the Heritage Foundation and The Wall Street Journal who jointly compile and publish the Index of Economic Freedom.

2.3.3 The Influence of DIME on the Operational Variables

Some research has been done to show that the instruments of national power influence the PMESII variables in measurable ways. Saie shows how a system of differential equations can be used to model the interactions between the PMESII variables and the instruments of national power over time (Saie, 2012: 34-55). Under the hypothesis that future system states are a function of the current state and exogenous forcing functions in the form of DIME inputs, he determines the relationship of the system states and the forcing functions using a non-linear least squares approach applied to existing, publicly-available data. He then solves a mean-field inverse problem to find the parameters of the differential equations. His results show that indeed, within the confines of the available data, the model can accurately predict the effect of DIME on the PMESII variables to a relatively high level of confidence (Saie, 2012: 56-89). Saie's research referenced previous work by Lanchester who first developed the idea of using differential equations to model attrition rates of two opposing forces, similar to the more commonly known predator-prey models that are detailed in Boccara (Boccara, 2010: 25-51).

2.4 Economic Warfare

The economic aspect of warfare has been recognized for centuries. Although not directly related to combat operations, military capability requires material resources, and the morale and will of a nation depends in large part on the ability of that nation to feed and house itself. In light of this, military action has been used throughout history to disable opponents' war-making abilities by destroying crops, transportation routes, and imposing blockades and sieges. In modern times, the rules of supply and demand have been used more preemptively, subtly, and to greater effect. Specifically, in limiting an enemy's supply of goods or funds many nations have employed embargoes on other nations, and others have nationalized foreign-owned businesses within their borders or seized financial accounts belonging to enemy entities.

Taillard develops a comprehensive outline of economic warfare. He begins by quoting Sun Tzu, who says "Those who render others' armies helpless without fighting are the best of all." He believes that increasingly more, modern warfare will evolve across domains until tools besides armed conflict are used more frequently, and to greater effect, than physical confrontation itself, which can be saved as a last resort. Chief among these tools is what he calls the "invisible fist of the market," or the wielding of economic power to strongly influence the actions of an enemy. He summarizes this view by saying "it is possible to force enemy combatants to surrender without a single physical engagement and … without the enemy [even] being sure whether any outside intervention has occurred..." (Taillard, 2012: 1-8) Among the strategies he details in his book are manipulating the supply of goods, whether through limiting supplies or creating excess supply (e.g., counterfeiting); manipulating trade; and manipulating markets.

Some of the tactics described by Taillard are already in wide use by world governments, while others are merely proposed. The concept that economic forces are used to subtly combat other nations is not new. The authors of *Unrestricted Warfare* mention specific instances they claim to be examples of financial warfare, or financial terrorism, and they assert that such tactics will be used in the future by one nation to disable another by crippling its economic and financial systems. Specifically, they name George Soros' successful currency trades against Britain and Malaysia as examples (Qiao & Wang, 1999: 6, 52-53). Soros' attack on the Bank of England serves as an illustration here.

In 1992 Soros famously "broke the Bank of England" by betting that the British pound and other European currencies were overvalued compared to the German deutsche mark. In their summary of the events, Jaffe and Machan explain that the then-extant European Exchange Rate Mechanism was created partially with the intent to keep the European currencies stable against each other. This was possible only if differences in interest rates and inflation rates among the 11 participating countries were checked by the participating central banks who were supposed to intervene in the currency markets and buy a weak currency to counter speculators and currency hedgers. Up until 1992, this strategy had worked. When the banks bought massive amounts of the weaker currency, the market was flooded with the stronger currency and the markets stabilized. In 1992 however, the strategy backfired when the Bank of England was forced by short sellers to withdraw from the Exchange Rate Mechanism. Speculators, observing that the pound was overvalued relative to other currencies, began dumping it and several other overvalued currencies such as the Italian lira. Soros and other short sellers were able to sell pounds faster than the British were able to buy them, thus causing a crash in the price of the pound. Eventually, the Bank of England was forced to lower its interest rate, stop buying the shorted pounds, and succumb to higher inflation. As a result of their action, Britain effectively withdrew from the Exchange Rate Mechanism, and Soros' group of traders made a windfall profit (Schaefer, Jaffe, & Machan, 2015).

While observers note that the overall effect of the speculators attack on the British pound was beneficial to Britain in the long run, and such an occurrence is unlikely to happen in the same way again soon, the potency of the traders' collective action is illustrative of the power that can be wielded by a determined player in financial markets.

2.5 Summary

Economic warfare has existed in some form for centuries, yet as the world evolves, it is gaining new prominence. As cited in Section 2.2, the Chinese military establishment is already considering offensive financial attacks, among other tactics, as potential weapons in future conflicts. Multiple non-state actors have already taken advantage of pricing disequilibriums within various financial markets, sometimes at the expense of entire nations and banking systems. With the number of cyber-attacks increasing and becoming an important element of warfare, it is very likely that these will combine with financial tactics to create havoc in American markets and harm the United States in non-military ways. Conversely, such attacks could be used by the United States to destroy another nation's ability to trade or raise money, and could preclude the necessity of armed intervention. An increased understanding of the dynamics of financial markets serves to accomplish both these purposes.

III. Literature Review

3.1 Chapter Overview

This chapter examines past work on the topic of complex adaptive systems, the application of differential equations to economic models, and the method of least squares used to solve inverse problems. However, it first reviews stock market dynamics and some of the prevailing theories that describe market behavior. Each of these topics are addressed separately, but together they provide background and justification for the methodology undertaken in Chapter IV.

3.2 Stock Market Dynamics & Modeling Techniques

Macroeconomic data is widely available through a variety of sources, but considering the hundreds of economic datasets that are available, some discrimination is necessary in variable selection. Multiple sources indicate that the S&P 500 is widely accepted as the best measure of the overall stock market because it captures approximately 80% of total stock market value (Bodie, Kane, & Marcus, 2004; S&P Dow Jones Indices, 2015a; Zoll, 2012). Thus, it is frequently used as a proxy for the overall U.S. stock market, and is included in the model developed in Chapter IV.

The market determines stock prices as a result of supply and demand for shares. By extension, the level of the S&P 500 Index is set by the supply and demand for shares of its constituent firms. The S&P 500 is a market value-weighted index, and the percent change in the total market value of firms in the index corresponds directly to the percent change in the index level itself. Market value is calculated by multiplying the number of outstanding shares for each firm by the firms' respective market price, then summing these values across all 500 firms in the index. Most stock pricing models, including the capital asset pricing model (CAPM), assume individual investors cannot affect prices by their individual trades, analogous to the perfect competition assumption of microeconomics. Thus, it is assumed stock equilibrium prices are set based on the demand of all investors in the market for a particular security. Furthermore, investors' demand for multiple stocks is obtained via "horizontal aggregation" or rather, the sum of investor demand across all securities in the market, or in this case, in the S&P 500 (Bodie *et al*., 2004).

The aggregate demand for stocks changes constantly as a function of the stocks' expected return, assessment of risk, current price, and the return and risk information of alternative investments (Bodie *et al*., 2004). The supply of stock shares on the other hand, is fixed at any given instant, but does change intermittently due to stock splits and firms repurchasing shares. Thus, assuming perfect competition and no transaction fees, the equilibrium price is always located at the intersection of the supply and demand curves. A notional depiction of the supply and demand for stocks is shown in Figure 3.1. .

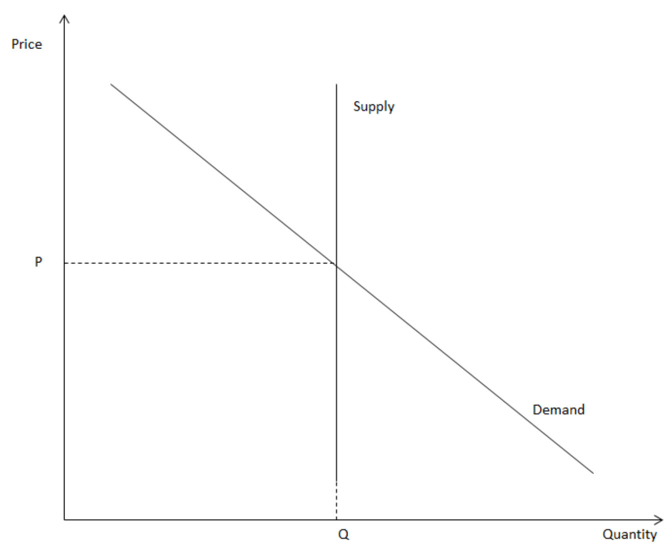


Figure 3.1: Equilibrium of supply and demand

Bodie explains that since a market index of stocks like the S&P 500 has a nearly perfect correlation to the entire market and is completely diversified, it has almost no non-systematic risk, which is the risk inherent to individual stocks. It is only subject to the risk present in the economy itself. Furthermore, investors require compensation for exposure to systematic, or economic risk, in the form of a "risk premium" that comes from the expected return of the stocks (Bodie *et al*., 2004). Thus, as stock returns and systematic risks change, investors' demand for stocks also change. Going one step further, as the demand for stocks shifts, assuming the supply of shares remains fixed, the equilibrium price for stocks also adjusts. Likewise, if the supply curve shifts due to changes in the number of outstanding shares, particularly in the case of stock buybacks, the equilibrium price changes accordingly.

Stock returns are calculated as the combined increase in dividends and prices over time, and investors can profit from both if they buy the asset below the future value-adjusted price. It follows that stock prices increase in anticipation of future returns until a new equilibrium point is reached. This is consistent with the Efficient Market Hypothesis (EMH) articulated by Fama, in which market prices respond quickly to any new information regarding stock returns (Fama, 1970: 383-417). While increases in stock prices come principally as a result of companies' earnings announcements or estimates, the announcement of dividends themselves increase a firm's future dividend yield and make the stock more valuable to investors (Bodie *et al*., 2004).

Company earnings are used in three primary ways. They can be retained by the firm to facilitate future growth, they can be distributed to shareholders as dividends, or they can be issued to investors in the form of share buybacks that decrease the supply of outstanding shares, increase the stock's equilibrium price, and result in capital gains for the investors (Bodie *et al*., 2004). In the case of increased dividends and buybacks, higher earnings result in immediate stock price increases due to the direct short-term returns in investor income or stock value. Retained earnings, on the other hand, can have a delayed effect on stock prices, depending on how they are used.

Business earnings are reported on a quarterly basis, and each quarter a firm's board of directors must decide how much earnings to allocate to shareholders or retain within the company. Sometimes, retained earnings are distributed to shareholders during a later quarter, but if a company determines the present value of future earnings will exceed current returns, the earnings will be reinvested into the business. This occurs often in rapidly growing firms, and their stock prices adjust upward as long as earnings continue to increase, or are expected to increase by investors who want higher returns (Bodie et al., 2004).

Thus, higher earnings have a profound effect on stock pricing, and firms' decisions regarding how to use the earnings can have strong impacts on prices. When aggregated across the S&P 500 index, earnings, dividends, buybacks, and retained earnings directly affect the price valuation of the 500 companies in the index, and therefore on the index itself. Numerous studies, including those of Fama, Sharpe, and Malkiel, support valuing stocks based on their expected returns and risk.

As is evident from any study of stock market behavior, many other forces exhibit influences that are not readily explained by the prevailing theories. Extreme market fluctuations are not modeled in the EMH since the EMH asserts that market fluctuations are normally distributed. According to this view, large deviations are statistically located in the tails of the distribution, and they cannot be attributed to the regular business fundamentals that usually drive asset valuations (Malkiel, 2003: 75-76). While this assertion may be correct, the gravity of certain market movements has had a powerful influence on stock performance over the years.

Some research has been done to explain the large market crashes of 1929, 1987, 2000 and 2008. The crashes of 1987 and 2008 are of particular interest here because of their relatively recent occurrence and because of the suddenness of their onset. In retrospect, their development can be perceived as obvious, but in the moment, because of their very nature, they are difficult to detect and/or acknowledge (Browning, 2007).

A common explanation for extreme market movements is that investors respond to news information regarding market fundamentals. However, Cutler, Poterba, and Summers, with others, have found limited evidence to suggest this is the case (Cornell, 2013; Cutler, Poterba, & Summers, 1989: 4-5). By fitting vector autoregression models to macroeconomic variables that measure real and financial conditions from 1926 to 1985, they were able to identify the unexpected component of a set of time series stock returns. Their findings suggest that only one third of the variance in stock returns can be explained by available economic information, such as stock returns, inflation, and so forth, suggesting the remaining variance is explained by "informational freeloading" of observed asset prices (Cutler *et al*., 1989: 11). In other words, a majority of investors do not conduct their own fundamental analysis, and instead believe that asset prices represent true values. It follows then, that they are susceptible to responding to those prices en masse, even in the midst of a market crash. In their words:

> The possibility that many investors do not formulate their own estimates of fundamental value is consistent with trading patterns surrounding the sharp stock market decline of October 1987. Despite the market's dramatic drop, the vast majority of shares were not traded. This is only explicable if investors rely on market prices to gauge values, or if investors received information that led to significant downward revisions in fundamental values. It seems difficult to identify the information that would support the second conclusion (Cutler *et al*., 1989: 11).

More recently, in 2012 Cornell replicated the work of Cutler and found that

> despite the explosion in information technology, enhanced market regulation, innovation in stock trading and the introduction of new equity related financial products, large movements in the market remain as common and mysterious as ever…If anything, the mystery has deepened because the size of the unexplained market movements has grown" (Cornell, 2013: 7).

In light of this research, it seems that basic macroeconomic news and market information has a limited effect in terms of significant market movements.

Rather, it seems more likely that markets exhibit a form of herd behavior in the short term, in which investors base their decisions on those taken by a majority of investors. Like Cutler, Beachy points out that many economists have argued that investors do not primarily consider the underlying value of an asset in deciding their willingness to pay for the investment. Instead, they focus on whether the investments are increasing or decreasing in the market at large. Additionally, "humans tend to excessively depend on recent and relatively small samples of information to project future trends – a bias known as the representativeness heuristic"(Beachy, 2012: 9). Shefrin describes some of the psychological biases that beset financial institutions during the housing boom of the 2000's.

> …Overconfident Merrill Lynch executives sidelined their company's most experienced risk managers and proceeded to boost their company's exposure to subprime mortgages. Investment bankers at UBS were beset by confirmation errors, searching for evidence confirming their rosy assessments of the subprime markets and ignoring disconfirming evidence gathered by their own analysts. Analysts at the financial products division of AIG were misled by categorizing errors, effectively relegating to a single category the credit default swaps they were selling, ignoring differences in the subprime composition of mortgage pools. And executives at Standard and Poor's, aspiring to enhance their wealth and position, chose to lower their standards for rating mortgage securities rather than lose business to competitors (Shefrin & Statman, 2011: 4).

Such arguments are at the forefront of behavioral economics and contradict many assumptions of investors as rational and independent. They also help explain the occurrence of asset price bubbles and crashes.

3.3 Inverse Problems

Cox explains the inverse problem by first describing a “forward” or “direct” problem (Cox, Embree, & Hokanson, 2012:158-160). Given that we know the material properties of a system, the forward problem is solved by injecting a stimulus into the system, then measuring the resulting outcome. By so doing, a direct relationship between the inputs and outputs is observed and a model is formulated that can predict future outputs within the specified region. Yet, many systems, such as world economies, seismic activity, and biological systems, present the problem backwards. The outward effects of a stimulus can be measured, but the internal system mechanisms that digest the stimulus and output the response are unobservable. Thus, only select inputs and outputs can be measured without knowing the internal composition of the system itself. In other words, the functional form of the system is unknown, as are the parameters that describe the internal functions. The inverse problem therefore attempts to solve for these parameters by making use of the observable inputs and outputs within a conjectured model (Cox *et al*., 2012:157-158). The logical extension of such an exercise is that by accurately calculating the system parameters, future behavior can be predicted given certain inputs.

One famous application of the inverse problem was described by Kac in 1974 when he asked “Can you hear the shape of a drum?” He posited that while a listener may not be able to see a drum, he could, by solving the inverse problem, determine the shape of the drum based on the acoustic signatures of the sounds being produced (Kac, 1974:534-535). Cox asks a similar question by mimicking the work of Krein, who demonstrated that by measuring the acoustic signatures of a taut string, one can hear the mass distribution of beads placed at various locations on that string. To do this, they first solved the forward problem by threading a piano string through n beads and applying a known tension between two clamps set at a known distance apart. With this structure in place, they plucked the string and measured the displacement of the vibrating string. They point out that their resulting set of equations are linear, and when organized into matrix form, the matrices are conveniently symmetric and positive definite. Then, after formulating the differential equation, they show how the inverse problem can be solved using the eigenvectors of the input matrices (Cox *et al*., 2012:158-161).

Saie's research attempted to solve a different form of inverse problem. That is, he attempted to model the relationship present between data on the PMESII variables discussed in Chapter II, and the data on DIME exogenous variables using differential equations. However, he did not know the parameter values to use in the model, even if the outward relationships between the datasets were readily apparent. Thus, he solved the system of differential equations to obtain the parameters, thereby solving an inverse problem (Saie, 2012:34-54).

Gomez-Ramirez emphasizes the need to employ inverse problems and methods to solve economic problems. Modern economies, with their manifold, powerful, and independent agents – all exhibiting human emotions, behaviors, and biases are one of the most complex and adaptive systems known. While various economic laws and principles have been discovered, much work remains to be done in identifying the causes of economic phenomena (Gomez-Ramirez, 2013:1-13). For example, since the late 1950's, the Phillips Curve has been used to describe the relationship between inflation and unemployment. Empirical evidence suggests an inverse relationship between the two, but critics such as Milton Friedman have contended that the relationship is only valid in the short run, and it cannot be used as an explanation for long run behavior or as an indicator of future trends (Novotná, n.d.:78-79). Despite these criticisms however, it is still used at the Federal Reserve to inform monetary policy because no better tool has yet been devised.

Similarly, while it is known that domestic stock prices are dictated largely by the supply of shares and the demand from potential investors, empirical evidence also shows that demand for stocks is affected either directly or indirectly by macroeconomic variables such as interest rates and the performance of alternative investments such as bonds, real estate, and foreign equities. Likewise, the supply of floating shares is affected in the aggregate by companies' decisions to distribute earnings via dividends or stock repurchases. Stock repurchases, or "buybacks," in particular have had a strong impact on valuations, and some economists have even labeled the practice as price manipulation (Lazonick, 2015; Mason, 2015:1-38). These economic processes are inverse problems by nature. While certain inputs are known and measurable, the effect of these inputs in generating measurable outputs is largely unknown, especially when applied across the entire stock market.

3.4 Nonlinear Optimization and Model Fitting via Least Squares

Nonlinear programming assists in solving inverse problems. When a problem is non-convex or includes nonlinear terms in its objective function or constraints it is considered to be a nonlinear programming problem and can be solved by minimizing or maximizing the objective function. To do this, the algorithm used solves the problem by iteratively moving across the surface of the problem's polytope surface, searching for the lowest local minima, as shown notionally in Figure 3.2. While a global minimum always exists, this solution often can only be found given infinite time and resources. Instead, most solver algorithms, including many of those found in commercial software, use certain stopping criteria to decide when a current solution is sufficient. These criteria usually include a maximum number of iterations, a maximum solving time, or convergence criteria (Bazaara, Sherali, & Shetty, 1993:1-5; "Chapter 13: Nonlinear Programming," n.d.:410-411).

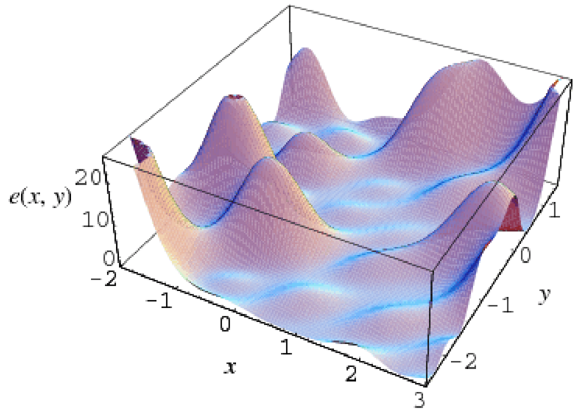


Figure 3.2: Example of nonlinear polytope surface. (Pintér, 2016)**.**

The general form of a nonlinear program is given in equations (3.1) through (3.4), where $f(x)$ is the objective function, and $g_i(x)$ is an inequality constraint, and $h_i(x)$ is an equality constraint. The problem is considered nonlinear if the objective function or any of the constraints are nonlinear.

$$\begin{array}{lll} \text{Minimize} & f(x) & \quad (3.1) \\ \text{subject to} & g_i(x) \le 0 \quad \text{for } i = 1,\ldots,m & \quad (3.2) \\ & h_i(x) = 0 \quad \text{for } i = 1,\ldots,l & \quad (3.3) \\ & x \in X & \quad (3.4) \end{array}$$

In this formulation, the functions f, g_i and h_i are defined on R^n, $X \subset R^n$, and x is a vector of n components $x_1,\ldots,x_n$. To solve the problem, values are found for the decision variables $x_1,\ldots,x_n$ such that the objective function is minimized while satisfying the constraints (Bazaara *et al.*, 1993; Saie, 2012).

Bazaraa explains that a vector $x \in X$ is a feasible solution if it satisfies all the constraints. The collection of all feasible solutions is referred to as the feasible region. The purpose of the nonlinear program is to find a feasible point $\bar{x}$ such that $f(x) \geq f(\bar{x})$. This point is the optimal solution. Multiple optima may exist however, consisting of alternative optimal solutions (Bazaara *et al.*, 1993:1-5).

3.4.1 Model Fitting via Least Squares

Boyd and Vandenberghe describe the least squares problem as a special case of convex optimization where the objective function is the quadratic sum of squares term, initially having the form $a_i' x - b_i$, as shown in equation (3.5). The sum of squares of this term is obtained from the squared ℓ^2 norm of the quantity, $Ax - b$, or $\|Ax - b\|_2^2$. Here, $A \in R^{k \times n}$, with $k \geq n$, and a_i' are the rows of A, and the vector $x \in R^n$ is the decision variable (Boyd & Vandenberghe, 2004:4-5).

$$\text{minimize } f_0(x) = \|Ax - b\|_2^2 = \sum_{i=1}^{k} \left(a_i' x - b_i\right)^2 \tag{3.5}$$

They continue by explaining that the solution of a least squares program can be reduced to solving a set of linear equations of the form $(A'A)x = A'b$, which gives the analytical solution $x = (A'A)^{-1} A'b$. Modern computing and highly accurate and reliable algorithms make least squares problems comparatively easy to solve, in a time approximately proportional to $n^2 k$. In practice, problems consisting of tens of thousands of variables can rapidly be solved using only desktop computers. Hence, the process for solving most least-squares problems is considered a mature technology (Boyd & Vandenberghe, 2004:4-5).

The method of least squares forms the foundation for regression analysis and many other parameter estimation and data fitting methods; recognizing an optimization problem as a least squares problem is straightforward (Boyd & Vandenberghe, 2004:4). Solving the inverse problem naturally requires estimating parameters in order to fit the results of a conjectured model to existing data. Once the model is formulated and data points predicted, the distance between these data points and the actual data is squared and summed, forming a quadratic function that is positive semidefinite (Boyd & Vandenberghe, 2004:5).

Weighted least squares is a modified version of the simple equation presented in (3.5). This form of the objective function includes a positive weight, w_i, for every set of actual and modeled data points where the distance is minimized, $i = 1, \ldots, k$. The weights are chosen to reflect different levels of importance on the sizes of $a_i{}'x - b$, or just to influence the final solution. This is a standard formulation of the least squares problem, and is commonly used to estimate the vector x when the linear measurements may be corrupted by unequal variances (Boyd & Vandenberghe, 2004:5). The general form of the weighted least squares model is shown in equation (3.6).

$$\text{minimize} \sum_{i=1}^{k} w_i \left(a_i{}'x - b_i\right)^2 \tag{3.6}$$

Nonlinear optimization has become increasingly important in engineering, finance, government, and many other industries as competition has increased and resources have become relatively scarcer. Rather than simply accept a usable design with large safety factors, optimization is used to ensure designs are the best possible (Bazaara *et al*., 1993:1). This principle certainly applies to math models of the sort addressed by this research. The viability of the model depends on the accuracy of the parameters chosen for it.

3.5 Complex Adaptive Systems in Economics

Until very recently, the application of complex systems has been reserved to the physical sciences, such as biology, ecology, chemistry, and physics. It has not been used extensively in the social sciences, and especially not in economics, which has long been dominated by "neoclassical" economic theories and tools. Foster points out that traditional economics has relied on models of constrained optimization in which economic agents, individuals or firms, behave rationally in trying to maximize their own gain within a set of constraints at a certain point in time. He faults this perspective as overly "simplistic" and inherently flawed since it attempts to explain complicated economic phenomena as singular occurrences, independent of other components within the system. Usually, simplifying assumptions are applied in order to understand economic behavior individually, but more importantly, current microeconomic theories do not consider the time-based evolution of interconnected behavior and thought that created the phenomena in the first place (Foster, 2004a, 2004b). Thus, Foster argues that economies and economic systems should be treated as complex adaptive systems.

Foster and others argue that economies should be modeled as networks that evolve and adapt over time on multiple levels ranging from fund flows to the exchange of ideas.

> The appropriate construct to understand systems at all levels is the network. The brain is a network, consumption spending lies in a network of interconnected tastes and interconnected income flows, production is a network, the whole economy is a network. Think of the firm, it is a network and, although firms' networks are similar in many respects because of the presence of higher networks, ie, a state space does exist, they differ in terms of the completeness, strength and particular qualities of their network structures. It is this that determines if a firm can generate value that yields a profit. This value does not just come from the elements contained in the firm – the individuals, the machines,etc. – but from the connections that are forged between them (Foster, 2004a:17).

Foster asserts, the connections at all levels of an economy are what make the economy function. It is not the independent, profit-maximizing behavior of individuals and firms that determines an economy's output, but rather the fact that the producers are creating goods with a designated purpose in mind, namely that they can be sold to consumers who have similar desires for the good. It is the connectedness of the components of the economy that makes the whole system function, and, at a higher level, the innovation and creativity of producers and consumers at all levels makes the economic system adaptive. As the system adapts to new goods, opportunities, threats, and circumstances, the collective memory of the economic system over the time dimension ensures that evolution occurs (Foster, 2004a:16-23).

Along these same lines, Levin defines a complex system as one that can absorb information from its environment and create stores of knowledge to aid action. He states that complex adaptive systems have three properties: diversity and individuality of components, localized interactions among the components, and an autonomous process that uses the outcomes of those interactions to select a subset of those components for replication or enhancement (Levin, 2002:4).

Foster defends his proposal of economics as a complex science by citing suggestions in the work of early-20th century economists, when the science of complex systems was still nascent. He believes these economists had an intuitive sense of the complex adaptive behavior at work in the economy. He mentions the work of Keynes and Schumpeter in particular, stating that they understood the macroeconomy as a system of interacting but autonomous components, and as these components exert their influence in both complementary and competitive ways, the system evolves and adapts over time (Foster, 2004b:32).

3.6 Economies as Dynamical Systems

Nonlinear dynamics of the form $\dot{x} = f(x,t)$ have been used to study changing economic variables for centuries. Magistretti points out their use in understanding economic growth, cycles, and market analysis. These models however have typically been simplified to linear approximations in order to permit their study. In recent years, due to the advent of modern computers, we are better able to study the original models as accurate depictions of reality (Magistretti, n.d.).

Magistretti provides an example of how income affects population growth in the Malthusian model. In this example, the logistic population growth model

$$\dot{N}(t) = aN(t)\left(1 - bN(t)\right), \tag{3.7}$$

which accounts for resource effects on population growth, $N(t)$, becomes

$$\dot{N}(t) = aN(t)\left(1 - b\frac{N(t)}{Y(t)}\right), \tag{3.8}$$

in the Malthusian model by dividing the second $N(t)$ term by $Y(t)$, the total income of the population (Magistretti, n.d.:1). He then shows how a dynamic nonlinear model of similar form can be used to depict the behavior of monopolies and duopolies. He also analyzes some characteristics of these models, such as the bifurcation points and equilibrium points of the model.

Montero used a dynamic model more specifically in modeling the price evolution of financial assets. He first used agent-based modeling to depict the stock traders in a market, then he derived a master equation that characterized the time evolution of the system and analyzed the stationary solutions of the equation (Montero, 2009).

Novotna used differential equations to show the relationship between unemployment and inflation. The Phillips curve was introduced by A.W.H. Phillips in the mid-twentieth century and describes the inverse relationship between inflation and unemployment. His studies found that when unemployment was high, wages increased slowly; but when unemployment was low, wages rose quickly (Hoover, 2008; Phillips, 1958). Novotna hypothesized that the relationship between inflation and unemployment could be modeled using a system of delay differential equations, which are differential equations where the derivatives at the current time depend on the solution, and possibly its derivatives, at previous times (Kuang, 2012:163-166). She used two endogenous variables, π and p, for inflation and expected inflation, respectively, and one exogenous variable, u, for unemployment. Then she added m to denote the growth rate of the money supply and created a system of two linear differential equations,

$$\begin{aligned}\frac{dx_1(t)}{dt} &= a_{11}x_1(t) + a_{12}x_2(t) + b_{11}x_1(t-1) + b_{12}x_2(t-1) + q_1\\ \frac{dx_2(t)}{dt} &= a_{21}x_1(t) + a_{22}x_2(t) + b_{21}x_1(t-1) + b_{22}x_2(t-1) + q_2\end{aligned} \tag{3.9}$$

or alternatively,

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t) + q \tag{3.10}$$

where $A = \begin{pmatrix} 0 & -j\beta \\ k & 0 \end{pmatrix}$, $B = \begin{pmatrix} j\alpha & j\beta\eta \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} -j(\beta z + p) \\ -km \end{pmatrix}$, and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $t \in [0, T]$. As given here, j and k are scalars, and α, β, and η are coefficients. The variable z represents various "microeconomic determinants," and the x variables are inputs to the system at time t.

Novotna argues that this model more accurately represents real economic situations because it models the changes in unemployment and inflation as functions in time, respecting the influence of the factors in the past (Novotná, n.d.:79-81).

3.7 Complex Adaptive Behavior in Financial Markets

Other authors have noted complex adaptive behavior in financial markets or explained significant market movements in ways that illustrate underlying dynamical systems. Scheffer identifies complex adaptive behavior as the reason why predicting movement in financial markets is so difficult. As soon as researchers or investors identify a pattern in market behavior, investors devise a way to profit from the pattern, thereby eliminating it (Scheffer *et al*., 2009:57). Scheffer claims that many complex dynamical systems, including financial markets, have critical thresholds, or "tipping points," at which a system abruptly shifts from one state to another. These critical thresholds are frequently identified by a characteristic "slowing down" in perturbations just before the threshold is reached. They demonstrate this phenomenon mathematically using a simple dynamical system where the critical threshold is reached at a bifurcation point. In the model, γ is a positive scaling factor while a and b are coefficients.

$$\frac{dx}{dt} = \gamma(x-a)(x-b) \tag{3.11}$$

In this model, the rate of recovery after a small perturbation is reduced and approaches zero when the system moves toward a catastrophic bifurcation point. They explain that this model has two equilibria, $\bar{x}_1 = a$ and $\bar{x}_2 = b$, of which one is stable and the other is unstable. If the value of parameter a is equal to b, the equilibria collide and the bifurcation point is reached. To illustrate this, if $\bar{x}_1$ is assumed to be the stable equilibrium, we can see what happens when the equilibrium is perturbed slightly by adding the term ε.

$$x = \bar{x}_1 + \varepsilon \tag{3.12}$$

$$\frac{d(\bar{x}_1 + \varepsilon)}{dt} = f(\bar{x}_1 + \varepsilon) \tag{3.13}$$

Then linearizing the equation using a first-order Taylor series expansion and simplifying it:

$$\frac{d(\bar{x}_1 + \varepsilon)}{dt} = f(\bar{x}_1 + \varepsilon) \approx f(\bar{x}_1) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_1} \varepsilon \tag{3.14}$$

$$f(\bar{x}_1) + \frac{d\varepsilon}{dt} = f(\bar{x}_1) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_1} \varepsilon \rightarrow \frac{d\varepsilon}{dt} = \lambda_1 \varepsilon \tag{3.15}$$

we can find the eigenvalue representing the rate of recovery from the perturbation:

$$\lambda_1 = \left.\frac{\partial f}{\partial x}\right|_{a} = -\gamma(b-a)$$

and

$$\lambda_2 = \left.\frac{\partial f}{\partial x}\right|_{b} = \gamma(b-a) .$$

Scheffer *et al*. explain that if $b > a$, then the first equilibrium has a negative eigenvalue and is thus stable because the perturbation goes exponentially to zero. Whereas, at the bifurcation point, $b = a$, the recovery rates λ_1 and λ_2 are both zero and the perturbations do not recover (Scheffer *et al*., 2009:55).

Bates adds credence to this theory and further describes complex system behavior in his article on the crash of 1987. He notes that around the time of the crash, there were no major economic developments that could explain the magnitude of the crash. Accordingly, he examines an alternative hypothesis that the crash "was a self-fulfilling prophecy – a 'rational bubble'." The reasoning here is that the fear of an expected crash is what sustains a bubble. To test whether this occurred, he examined the spread between call and put option prices on S&P 500 futures leading up to the crash to see how many investors were actually predicting the fall. Reviewing data from 1985 to 1987, he shows that out-of-the-money puts became "unusually expensive" in the year preceding the crash (Bates, 1991:1009). Since a call option only becomes valuable when the underlying asset's price exceeds the option's exercise price, and puts are valuable only when their exercise price exceeds the asset's price, if investors are predicting a sudden market decline, put options on S&P 500 futures with exercise prices below the current futures prices should be priced higher than calls with exercise prices above the futures price. This occurs because the perceived likelihood of large downward movements in the market makes the puts more likely to finish in the money than the calls, thus making them more valuable and more in-demand by investors, which in turn raises the price on those options. His results indicate that indeed, investors did fear a crash in 1987, evidenced by out-of-the-money put options on S&P 500 futures becoming more expensive relative to out-of-the-money calls. Yet, the crash fears peaked in August and subsided thereafter until the actual crash occurred in October (Bates, 1991:1010-1012). Thus, the pattern of perturbations in options prices were seen to increase, then "slow down" until the critical threshold of a crash occurred on 19 October 1987.

Yalamova and McKelvey further this perspective on market crashes by drawing analogies from physics. They assert that there is significant evidence to support both the Efficient Market Hypothesis (EMH) and more dynamic theories of market behavior. In explaining this view, they create a "phase-transition model" depicted graphically as a two dimensional diagram separated into three regions. Each region represents the actions which investors can take. The first is "Wait," which is what investors choose to do when stock prices are in equilibrium and supply equals demand. The second region is "Buy," representing investors' choice when securities are undervalued. The third region is "Sell," which occurs when assets are overpriced. On average, and consistent with the EMH, the market is in equilibrium at or near the intersection of these three regions. Equilibrium is maintained because any small arbitrage opportunities are quickly acted upon by independent, enterprising investors with information that is "heterogeneous" to that of other traders. However, as these "noise traders" identify winning strategies, small groups of traders begin to lose their heterogeneity of information by trying to replicate the winners' strategies(Yalamova & McKelvey, 2011). As this occurs,

> …prices destabilize and periodic orbits emerge as demand distribution bifurcates and imitation and information cascade amplify… Once traders lose their heterogeneity by learning from each other and from market results what the best rules and formulas appear to be, what began as a "tiny initiating event" in the form of an experimental new investment strategy spirals up into a widespread belief about how best to win out over noise and take increasingly leveraged but seemingly well-defined risks. Convergence of thousands of traders on a particular formula – what is really convergence toward a single buy-sell rule – for some period of time sets the bubble-creation process in motion until some intervening event disrupts it and the market may not reach the Critical Point. Growing clusters of imitation and herding among market participants create a regime of trading synchronization exhibiting log-periodic oscillations in index levels… At the Critical Point "sell" orders prevail precipitating a market crash (Yalamova & McKelvey, 2011:179).

They conclude by stating that the preceding model can be applied on a larger economic scale, and the behavior indicating convergence of trading rules and homogeneous information was observed prior to the financial crisis of 2007-2010. In their view looking back, the impending outcome of this behavior should have been obvious to experts and leaders at the time, and the crisis should never have occurred (Yalamova & McKelvey, 2011).

Other, more general indicators have been identified as predictors of a market crash. As shown in the Yalamova and McKelvey model, the housing crisis of the late 2000's serves as a specific example of how a market bubble can form and then rapidly deflate, but the lessons learned from it can be generalized to apply to financial markets. Beachy names three principal causes for the housing bubble:

1. A large price bubble – an artificial and steep rise and fall in the price of a particular good or asset type;
2. A credit boom that exacerbates the bubble's size and impact, and
3. Skewed financial sector incentives that feed the credit boom (Beachy, 2012:7).

In explaining these causes, he notes that the housing bubble began from a real increase in demand for homes during the 1990s, created, in part, due to the wealth effect generated as a result of rising stock market values and the dot-com bubble. The resulting rise in prices then spurred speculators to buy homes in anticipation of future price growth, and this increase in demand drove prices even higher. Then, easy lending standards and low interest rates instituted in response to the mild recession of 2001 (caused in part from the deflation of the dot-com bubble) provided funds to homebuyers and speculators, thereby enabling them to buy more homes at ever higher prices. The availability of credit was fueled further by the rise in mortgage-backed securities as a profitable investment, thus providing incentives to banks and investment firms to continue offering funds at lower interest rates and at decreased lending standards (Beachy, 2012:7-8).

Beachy continues by saying that the relationship between credit availability and price bubbles has been seen numerous times throughout history. He references Gourinchas who writes that "a key precursor to twentieth-century financial crises in emerging and advanced economies alike was the rapid buildup of leverage... Data suggests that domestic credit expansion and real currency appreciation have been the most robust and significant predictors of financial crises…" (Gourinchas & Obstfeld, 2011). Later, Gourinchas names two underlying factors for financial crises, especially among first-world countries in the last few decades. These are "a buildup of domestic and external leverage in a context of explicit or implicit government guarantees to a liberalized financial sector, and real currency appreciation" (Gourinchas & Obstfeld, 2011:4). The implied warnings of the Gourinchas paper are especially salient to the United States in 2015. Bloomberg reported in March 2015 that the U.S. dollar had increased 24% since June 2014, while the explicit and implied government guarantees to financial institutions considered "too big to fail" have increased significantly since the financial crisis in 2008 (Chandra, 2015; Schich & Lindh, 2012:2). The most notable of these were the bailouts of Bear Stearns, AIG, and other financial institutions, as well as the $700 billion Troubled Asset Relief Program (TARP) during the Great Recession. The bailouts have raised concerns that with such implied guarantees, large banks will take on additional risk, which in combination with currency appreciation, exposes the financial system to the risk of another crisis (Beachy, 2012:45-51).

3.8 Summary

This chapter reviewed theories explaining stock market behavior and showed in the last section that highly liquid financial markets are dynamical by nature and behave as complex adaptive systems. In light of this evidence, some economists have argued in recent years that economics should be treated as a complex system science. They cite evidence that most parts of the economy are interconnected, but instead of acknowledging these evolving relationships, modern economic theory analyzes economic variables in static states.

Other literature was discussed that is relevant to the methodology in Chapter IV. Stock market dynamics were reviewed to identify important factors to include in the model, and inverse problems were examined in light of the fact that economics, as a science, deals extensively with "black boxes." To solve such problems, nonlinear optimization and the method of least squares is needed to discover the parameters that define the relationships between variables.

The following two chapters detail how these ideas are applied in this research. First, Chapter IV discusses the methodology used to create and solve the model. Chapter V then implements these methodologies and shows the results of the analysis.

IV. Methodology

4.1 Chapter Overview

This chapter details the methodology used to develop and solve the proposed model. It first discusses how the model was conjectured and introduces the model components as parts of a fully-connected network. Then, it reviews the procedures used to collect the data and normalize it for use in the model. Third, the network diagram of the model is translated into its functional form consisting of a system of differential equation. The reasoning behind the functional form is presented, as well as some of the implications associated with the model. Fourth, a methodology is created to solve the system of differential equations and determine its coefficients. Lastly, the chapter discusses how the model and its results will be analyzed. The steps of this methodology are shown graphically in Figure 4.1.

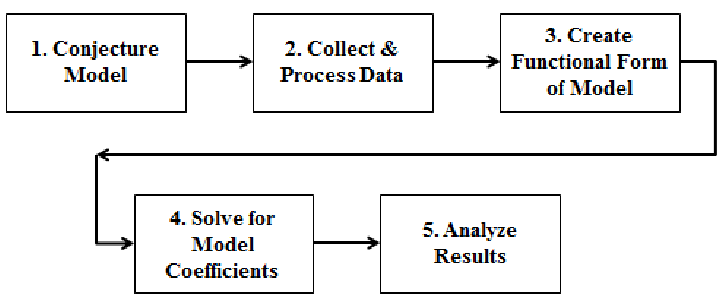


Figure 4.1: Methodology

4.2 Conjecturing the Model

Models are conjectured via one of two methods: a data-driven approach or a model-driven approach. A data-driven model is typical to regression techniques and seeks to determine relationships between a dependent response variable and independent regressor variables. These relationships can be identified through multivariate analysis that uses covariance and correlation matrices to find significant interactions between the variables. Stepwise model development is also used in which multiple regression models are evaluated iteratively by adding and deleting regressors until the best fit is obtained. These techniques rely primarily on first order interactions, although some higher-order interactions may be significant to the model. Data-driven analysis is valuable because it can be used to identify hidden factors that contribute to particular model outcomes. Assuming the correct set of data is available, the key factors of a system can be identified and the best possible model created to accurately replicate the real world.

Data-driven approaches can be problematic, however, when too much data is available or the underlying system displays dynamic behavior. In the case of the stock market and the U.S. economy, thousands of datasets exist that may contribute to stock market performance; and as time-series data in a dynamic environment, one factor may have a powerful effect on the market during a certain period of time, but exert less influence at another time. In these situations, a data-driven model is hard to implement, and relationships are difficult to determine.

In contrast, a model-driven approach conjectures a model and its relationships first, based on input from subject matter experts and observation. It then collects and applies the relevant data to the model and evaluates the accuracy of the model using metrics from a least squares approach, particularly the mean square error. Modeling the stock market as part of a complex adaptive system is better suited to the model-driven approach.

Section 3.5 referenced the work of Foster, who advocates for studying economics as a complex systems science and modeling economic systems as networks. Along these same lines, the interconnected nature of the U.S. economy was discussed throughout Chapter III. Conjecturing a model of the stock market and the factors that contribute to its performance is therefore conducive to a network diagram for describing its relationships; but first, the components that constitute the network must be articulated.

Chapter III evaluated the economic variables considered to exert key influences on the stock market. Section 3.2 reviewed literature that pointed to the S&P 500 as the best index for representing the overall American stock market. Later, company earnings were described to be indicators of future stock returns in the form of dividends and buybacks. Since earnings can also be used to fuel future company growth, retained earnings play a role in determining stock valuations and are inversely proportional to the capital expended by firms on dividends and buybacks. When each of these measures is aggregated across all the companies in the S&P 500, they serve as factors that affect the S&P 500 price level.

Based on the information presented in Chapter III, the S&P 500 price level, earnings, dividends, buybacks, and retained quarterly earnings, are all included in the conjectured model and shown in the network diagram depicted in Figure 4.2. Since the model is based on causal relationships, GDP Per Capita is added to account for changes in company earnings. Increases in the U.S. economy are measured by GDP, and these changes will transfer to average business earnings.

The last component of the model is the federal funds rate, which serves as the single forcing function in the diagram. This is logical since the federal funds rate is the principal and most flexible instrument available to economic policy makers, and it is included in the model to account for initial changes in GDP, earnings, and company decisions regarding dividends and buybacks.

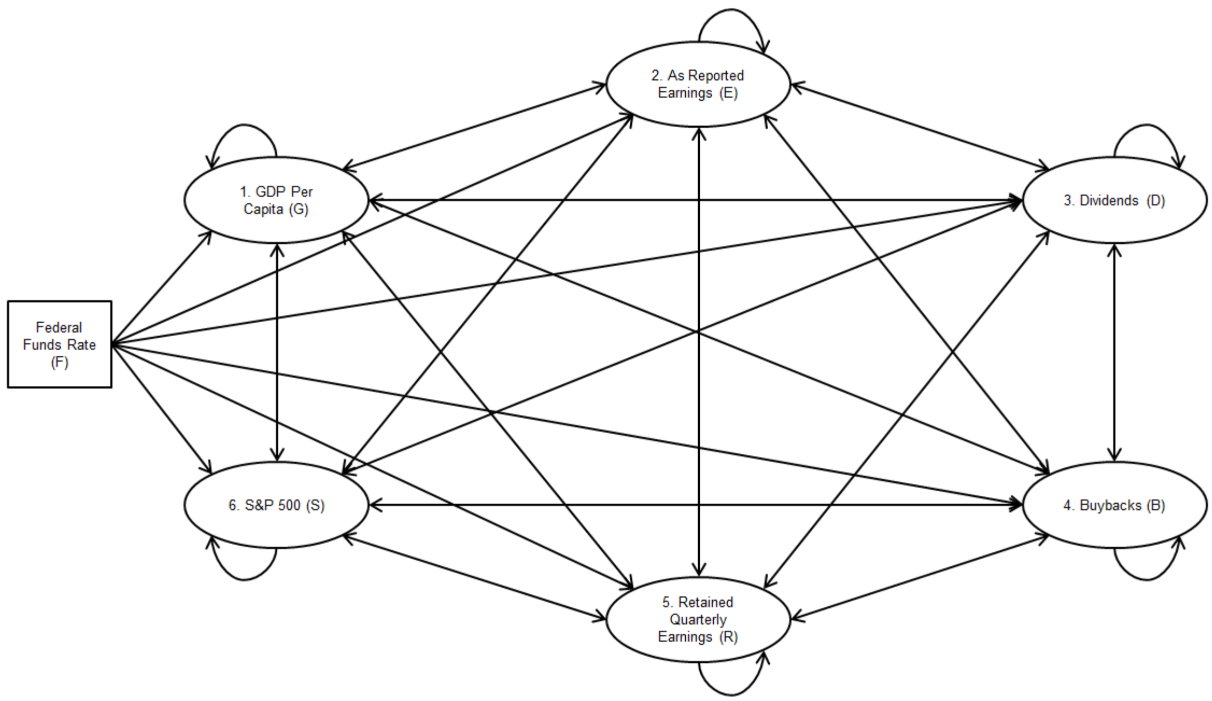


Figure 4.2: Model as a fully connected network with one exogenous factor

4.2.1 Model Assumptions

The structure of the model network rests on several assumptions and tries to capture key elements contributing to stock market activity. It first assumes that the American economy can be represented by the productivity and spending of its population. It further assumes that higher consumer spending and business production, as measured by U.S. GDP, will normally result in higher corporate profits. Thus, changes in GDP will affect corporate earnings, and particularly those of the companies that make up the S&P 500. When these earnings change, the companies in the S&P 500 must decide how to spend their earnings, given their current state of information regarding future economic growth and their desire to reward shareholders. These decisions determine how much earnings from each quarter will be retained to facilitate future growth, how much will be distributed in dividends, and how much will be spent to repurchase company stock.

In turn, stock market prices increase, decrease, or stay the same based on investors' reactions to business earnings, dividends, and buybacks, or the anticipation of these. Finally, monetary policy, represented by the federal funds rate, affects the availability of capital to businesses, thus changing their ability to finance business expansion or use debt to reward shareholders. It also affects consumer access to credit which could lead to higher GDP and increased business earnings. To a lesser degree, the federal funds rate may affect the stock market directly by changing the amount of capital available to investors themselves.

The model is represented as a fully connected network in which each of the six components affects itself and the other five variables. The federal funds rate, in turn, has an effect on each of the six components, and its level is determined independently by the Federal Reserve in order to influence current and future economic conditions. When the functional form of the model is presented in Section 4.4, each component of the model is represented by a letter, as shown in the ovals of the network in Figure 4.2. For example, GDP Per Capita is represented by the letter "G," As Reported Earnings by the letter "E," Dividends by "D," and so forth. Additionally, each model component is assigned a number in the ovals of Figure 4.2. GDP Per Capita is assigned the number "1," As Reported Earnings has the number "2," etc. This numbering system will be used in the functional form of the model as well, and throughout this research, as superscripts or subscripts. For example $x_t^{(i)}, i = 1, 2, ..., 7$ represents the value of model component i at time t, where $i = 1$ represents GDP, $i = 2$ represents As Reported Earnings, and so forth.

4.2.2 Model Component Data

A description of each component is provided next, detailing its role and interactions in the model. The data for each component was obtained directly from the agency that compiles it.

All the data series except for Federal Funds Rate were adjusted for inflation, as will be discussed in Section 4.3. Additionally, the GDP data was seasonally adjusted by its source, the U.S. Bureau of Economic Analysis, to remove the average effect of variations that normally occur at about the same time and in about the same magnitude each year, such as when farm production falls each winter. By making seasonal adjustments, true cyclical and other short term changes in the data tend to stand out more clearly (U.S. Bureau of Economic Analysis, 2006:24).

(1) **Gross Domestic Product (GDP) Per Capita [G].** GDP Per Capita is the principal metric of average U.S. output per person in the United States. It represents the market value of all goods and services produced within the United States and is calculated by taking the sum of personal consumption expenditures, gross private domestic investment (i.e., business investment, not financial investments), net exports of goods and services (i.e., exports less imports), and government consumption expenditures and gross investment, then dividing the total by the country's population. GDP excludes the intermediate purchases of goods and services by business (U.S. Bureau of Economic Analysis, 2006:22).

As U.S. production and spending increases, the growth is captured by the GDP metric. In addition to the federal funds rate, the model accepts GDP as a primary stimulus. Hence, increases in per capita GDP translate to higher consumer and business spending, which lead to increases in company earnings, and consequentially, stock prices. Immigration and other changes in the U.S. population can impact business earnings and GDP due to higher/lower demand for goods and increases/decreases in employment rates. These changes are considered in the model by using GDP Per Capita rather than GDP alone.

(2) S&P 500 As Reported Earnings [E]. In general, as reported earnings refer to the after-tax net profit of a company. In the model, the as reported earnings are the sum of total net earnings and losses for all 500 companies included in the S&P 500. For an index of stocks, as reported earnings conveys the most accurate picture of real earnings for the component companies, as opposed to operating earnings which are more erratic and firm-specific (Krisiloff, 2014). As reported earnings are governed by generally accepted accounting principles (GAAP) and are used in three primary ways, as depicted on company balance sheets. They can be distributed to shareholders as dividends, they can be used to buy back company shares, or they are retained for future use by the company as investment, dividends, or buybacks. Investors respond to increased earnings and the anticipation of dividends and buybacks by purchasing more stock, thus causing stock prices to rise. Data for as reported earnings was obtained from Standard & Poor's.

(3) S&P 500 Dividends [D]. Earnings are most commonly distributed to shareholders as dividends. They are usually distributed quarterly or annually, and many companies try to maintain a consistent record of steady or increasing distributions. Some firms in the S&P 500 have consistently issued dividends for over 50 years, causing their stock to be highly desired by investors interested in generating reliable income over time. Changes in dividend amounts affect a stock's dividend yield, which cause the stock price to adjust accordingly as investors try to capitalize on increased yield or escape from diminished yield. The model includes dividends because of its ability to influence stock prices, return money to the economy to affect GDP and as reported earnings, and the tradeoff decision made between retained earnings, dividends, and stock buybacks. Data for dividends was obtained from Standard & Poor's.

(4) S&P 500 Buybacks [B]. Earnings can also be distributed to shareholders via share repurchases, or "buybacks." Rather than issuing dividends or retaining earnings, a company's board can authorize the repurchase of a set amount of shares within a certain timeframe. Earnings are then set aside toward this objective, and when the timing is judged to be optimal, the company buys a certain quantity of the predetermined number of shares on the open market. Historically, it is not uncommon for companies to buy back shares when stock levels are already high in an effort to push share prices even higher. Buybacks have the effect of decreasing the supply of company shares available to investors, thus causing the stock price and the earnings-per-share to increase. The model supposes companies' decisions regarding buybacks have a strong effect on S&P 500 price, while also affecting dividends, retained earnings, and to a lesser extent, GDP, and as reported earnings. Data for stock buybacks was obtained from Standard & Poor's.

(5) S&P 500 Retained Quarterly Earnings [R]. Businesses may choose to retain a portion of quarterly earnings to facilitate future growth. For companies that are growing rapidly, it is not uncommon for them to retain all earnings in order to expand quickly and gain as much market share as possible. In this case, investors will often still buy the stock in anticipation of future dividends or buybacks. Because retained earnings affect the amount of earnings that can be allocated to dividends and buybacks, its effect on them is obvious. The model also asserts that retained earnings influence GDP by increasing domestic production via reinvestment, and it is assumed to affect as reported earnings because of its potential to generate future profits. Retained quarterly earnings are calculated by taking the difference of quarterly as reported earnings, less dividends and buybacks.

(6) S&P 500 [S]. Stock market data is represented by the S&P 500, which is widely accepted as the best gauge of the large-cap stock market. Membership in the S&P 500 Index is reserved for the largest 500 publicly-traded companies in the United States by market capitalization, regardless of which market they trade under, and it is estimated that the index captures 80% of the total U.S. market capitalization ("S&P 500 ® Fact Sheet," 2015:1). Eligibility requirements dictate that a company's market capitalization be $5.3 billion or more, and at least 50% of shares must be available for trading. Furthermore, all companies in the index must have positive reported net earnings in the most recent quarter, and over the last four quarters together ("S&P 500 ® Fact Sheet," 2015:1). Historical price data for the S&P 500 was obtained from Yahoo! Finance for the years 1998 through 2015.

(7) Federal Funds Rate [F]. The federal funds rate is the rate at which depository institutions trade balances held at the Federal Reserve with each other overnight. The weighted average rate at which banks lend to each other is the effective federal funds rate, which is determined by the market, but it is actively influenced by the Federal Reserve through open market operations in which a target rate is sought. The Federal Open Market Committee meets eight times a year to determine the target rate. It then influences the rate through buying or selling government bonds in order to keep rates at or near the target. Consequentially, liquidity and lower interest rates are introduced into the market by purchasing an increased number of bonds, whereas higher interest rates and decreased liquidity are obtained by selling bonds. The target rate is set in pursuit of the Federal Reserve's dual mandate to keep inflation and unemployment at productive levels. Many other interest rates are influenced by the federal funds rate, such as consumer mortgages, small business loans, savings and certificates of deposit rates, and corporate bonds (Board of Governors of the Federal Reserve System, 2015a).

The federal funds rate is intended to indirectly stimulate GDP, but it also affects the availability of capital to businesses and consumers that can lead to increased earnings for businesses. Additionally, some companies use debt to fund dividend distributions or stock buybacks. In recent years, low interest rates have directly led to higher stock prices, despite disproportionately low increases in company earnings. Instead of borrowing money to fuel expansion and research, the debt has been used to reward stockholders via increased dividends and buybacks. Some economists and politicians have referred to rampant and largely unregulated stock buybacks as price manipulation (Lazonick, 2014). Data for the federal funds rate was obtained from the St. Louis Federal Reserve.

4.3 Data Collection and Processing

Quarterly data is used in the model in order to more accurately capture the long-term trends of economic and financial market behavior. In the case of GDP Per Capita, As Reported Earnings, Dividends, Buybacks, and Retained Quarterly Earnings, these variables are only released in quarterly increments. As for the S&P 500 index and the Federal Funds Rate, which change constantly, averaging these values over quarterly increments has the effect of approximating the true distribution of the data without having to deal with its full complexity, thus providing the mean-field behavior of the system (Opper & Saad, 2001:1).

4.3.1 Adjusting the Data for Inflation

Converting nominal dollar values to real values is a common practice in economics and finance, allowing monetary quantities from different time periods to be compared directly. Moreover, by converting historical dollar amounts to present values, it is easier to identify with the results (Nau, 2016).

All monetary data in the model is adjusted for inflation using the consumer price index (CPI) as compiled by the Bureau of Labor Statistics. It is accomplished via a two-step process that converts all dollar values to second quarter 2015 dollars. First, an adjustor for each period is found by dividing the current period's CPI measure by the CPI value for 2015. The nominal dollar value of the economic variable in question is then multiplied by its period's CPI adjustor to get the real value of the measure. As an example, the conversion of nominal S&P 500 values to real dollars is shown in equations (4.1) and (4.2). CPI values used in the model are provided in Appendix A.

$$\text{Adjustor}_t = \frac{\text{CPI}_t}{\text{CPI}_{2Q2015}} \tag{4.1}$$

$$\text{Real S\&P 500}_t = \text{Nominal S\&P 500}_t \times \text{Adjustor}_t \tag{4.2}$$

4.3.2 Normalizing the Data

The data series are further adjusted to account for large differences in magnitude. Of the seven data sets, five have measures in the hundreds of billions of dollars, while one reaches only thousands, and the Federal Funds Rate is given in percentages. To compare the data sets evenly, without applying undue weight to any particular variable in the nonlinear program during analysis, each data series is scaled to adjust its maximum value to 80. This value is chosen arbitrarily, but it has the effect of keeping most data points within a manageable and familiar range. The simple formula for scaling the data is given in equation (4.3), where $x_t^{(i)}$ is the unscaled variable for each model component i at each time period t, $\theta^{(i)}$ is a scalar, and $y_t^{(i)}$ is the scaled variable.

$$y_t^{(i)} = \theta^{(i)} x_t^{(i)},\ i = 1,\ldots,7, t = 1,2,\ldots \mathrm{T} \tag{4.3}$$

The value of $\theta^{(i)}$ is found by simply dividing 80 by the maximum $x_t^{(i)}$ value for each of the seven model components, as shown in equation (4.4).

$$\theta^{(i)} = \frac{80}{\max\left(x_t^{(i)}\right)},\ i = 1,\ldots,7, t = 1,2,\ldots \mathrm{T} \tag{4.4}$$

The parameter $\theta^{(i)}$ scales each data point in i via multiplication. If the data series ranges into the thousands or billions, $\theta^{(i)}$ adjusts each data point down to a value less than or equal to 80. If the maximum value of $x_t^{(i)}$ is small, as with the federal funds rate, each data point in the i series will be scaled up until the maximum value in the series is equal to 80. This formulation works for the selected data sets because the maximum value of each is known to be greater than zero. If this was not the case, and $\max\left(x_t^{(i)}\right)$ was equal to zero, mathematical programming would be necessary to solve for the value of $\theta^{(i)}$.

4.3.3 Calculating the Data Slopes and Secants

The method of least squares is introduced in Section 4.5 and is used to fit a curve to the data using three separate approaches: minimizing the squared distances between the predicted points and the actual data points; minimizing the squared differences between the value of the differential equations and the value of the slopes in the actual data points; and minimizing the squared differences between the secants of the fitted line and the secants of the actual data. The first approach is based on the traditional least squares approach, in which the predicted data points are fit to the actual data points. For this approach, no modification to the data is necessary besides that which was discussed in Sections 4.3.1 and 4.3.2. The second approach relies on data points that are chosen such that the differential equations used to predict these points are set equal to the vertical change between each set of corresponding adjacent points in the actual data set. The equation for calculating the vertical change, or slope, between the actual data points is given in equation (4.5). The formulas for calculating the differential equations will be explained in Section 4.4.3.

$$\text{Slope of curve } i \text{ at time } t = \frac{f^{(i)}(t) - f^{(i)}(t-1)}{t-(t-1)},\ i = 1, 2, \ldots, 6 \tag{4.5}$$

Similarly, the third approach is obtained based on fitting the predicted data points so the secants between the predicted points are equal to the secants of the actual data. Secants are defined as the average rate of change between two nonconsecutive points and are calculated in the same way as the slopes, except the step sizes are larger. The formula for calculating the secants is given in equation (4.6), where v is the step size used to calculate the secants for each of the six model components, i.

$$\text{Secant of curve } i \text{ at time } t = \frac{f^{(i)}(t+v) - f^{(i)}(t-v)}{(t+v)-(t-v)},\ i = 1, 2, \ldots, 6 \tag{4.6}$$

After the predicted curves have been calculated using each of these three approaches, the fit of the curves will be judged against the actual data, slopes, and secants using three methods of evaluation. Particularly, the mean square error (MSE) and the coefficient of determination, or R-squared value, will be computed to assess the fits. Additionally, the maximum squared error resulting from each approach will be evaluated for each of the model components. If the actual data value corresponding to this error falls within the 95% prediction interval of the predicted data value, the predicted curve will be considered a good approximation of the actual data, even in its worst case. The approach that produces the overall best fit according to these methods will then be used for further analysis.

4.4 Creating a Functional Form of the Model

The functional form of the model is based on that of similar models where a carrying capacity is embedded within a set of differential equations in order to describe the fluctuations between populations over time. Like the Malthusian model in equation (3.8), which showed the change in population growth rates as a function of the population's income, Boccara presented the logistic differential equations for a predator-prey model that included a carrying capacity based on available resources. In the differential equation for the prey population, denoted as $\dot{H}$ in equation (4.7), the current population of prey, H, is multiplied by a proportional growth rate, b, and is subject to the environment's carrying capacity, K, in the denominator. While the growth rate simulates how quickly the population will grow, the carrying capacity indicates that at some point there will not be enough resources in the ecosystem to support further population growth (Boccara, 2010:31).

$$\dot{H} = bH\left(1 - \frac{H}{K}\right) - sHP \tag{4.7}$$

The notion of a carrying capacity is consistent with the law of conservation of energy, which also applies to economic systems as shown by Magistretti (Jaynes, 1991; Magistretti, n.d.). In the case of the unemployment rate, for example, the number of unemployed individuals in the labor force can never be less than zero and it cannot be greater than 100%. Indeed, the unemployment rate cannot even persist at high rates for too long if an economy is to survive.

Similarly, by nature of its composition, the S&P 500 cannot decline to zero except in the highly unlikely situation that every publicly-traded company in the U.S. were to go out of business. Neither can it rise too high before investors will stop buying stocks due to excessively low yields, assuming earnings remain constant.

The functional form for the model has a structure analogous to that of the Malthusian population growth model and the predator-prey model. In this case, the growth rate in the logistic differential equation is represented by the coefficient a, and the carrying capacity K is replaced by the coefficient b. In contrast to the predator-prey model, whose data is normalized to a range of [0,1], the economic model is only scaled so the maximum value of each data series is equal to 80; and, whereas the population model subtracts the ratio $\frac{H}{K}$ from 1 to convey that a population cannot grow beyond its capacity, the economic components of the model tend to increase over time, though not strictly so. Hence, in the economic model 80 is subtracted from the ratio $\frac{x_t^{(j)}}{b}$ to show that each model component cannot grow beyond its capacity. Here, the term $x_t^{(j)}$ denotes the state, or magnitude, of component j at time t, where $j = 1, 2, ..., 6$.

Putting these pieces together creates a new functional form to describe the derivatives in the model at each time step, or the effect that each variable j has on each variable i. This form is applied to each of the six components in the model, $i = 1, 2, ..., 6$, and is arranged as shown in equation (4.8).

$$\text{Effect of variable } j \text{ on variable } i \text{ at time } t = a\left(\frac{x_t^{(j)}}{b} - 80\right) \tag{4.8}$$

In this construct, the coefficient a represents a scalar used to weight the contribution of each economic variable j on the predicted variable value i, where the superscripts $i, j = 1, 2, \ldots, 6$ represent the six primary model components, or GDP Per Capita, As Reported Earnings, and so forth. Later, the coefficient d will be added to the model to represent a scalar on the federal funds rate exogenous variable.

The coefficient b represents a tipping point at which each component in the model causes a change in the dynamics of the system, such as in the case of a bifurcation point (Boccara, 2010:87). With a saddle-node bifurcation, as depicted in Figure 4.3, a small perturbation in the economic system may have a significant impact on the predicted variable i and consequently on the entire system. For example, when b is approximately 1/80$^{\text{th}}$ the value of the economic variable j at time t, the effect of the variable is stable. Yet, if $b > \frac{x_t^{(j)}}{80}$, there is a diminishing effect on the predicted variable i. Likewise, if $b < \frac{x_t^{(j)}}{80}$, there is a magnifying effect (Saie, 2012:47-49).

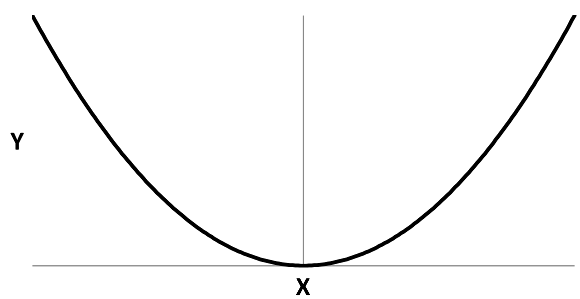


Figure 4.3: Saddle node bifurcation

The concept of a bifurcation point and importance weights for each economic variable illustrates the assumption that, like the predator-prey model and many other dynamic systems found in nature, economic systems have a certain growth limit, or carrying capacity, in the short term. While an economic variable may exhibit an exponential growth trend for a short amount of time, eventually market forces will temper the growth, causing the exponential curve to assume an “S” shape and plateau as time progresses, before either declining or rising further based on the inputs of the other variables. This behavior is apparent in many dynamical systems found throughout nature, including in the components of the model, but for illustrative purposes it is demonstrated here by the U.S. unemployment rate, charted over the period of 1958 to 2015. In Figure 4.4 the effect of bifurcation points becomes evident, resulting in the cyclical pattern of unemployment that also coincides with the regular business cycles in the United States.

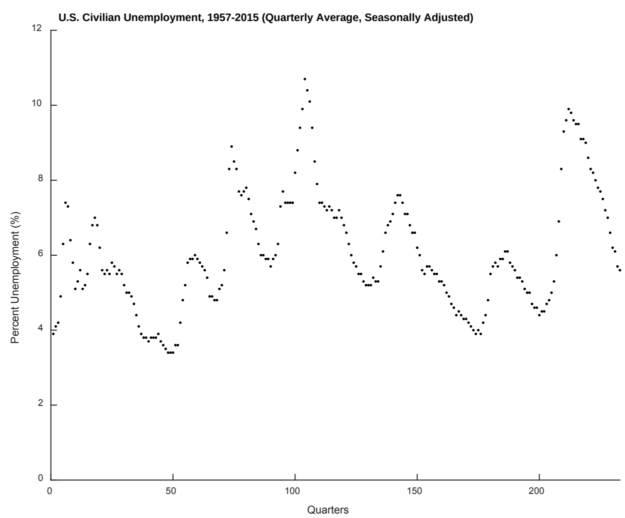


Figure 4.4: U.S. unemployment from 1957 to 2015

For the purposes of modeling a carrying capacity, or tipping point, the coefficient b in the model must have a defined range restricted to the interval (0,1]. Since a and d are only scalars, no similar restriction is necessary for them. Stated concisely, the ranges on the coefficients are

$$\begin{array}{ll} a_{ij} \in R & \text{for } i, j = 1,2,\dots,6 \\ 0 < b_{ij} \le 1 & \text{for } i, j = 1,2,\dots,6 \\ d_i \in R & \text{for } i = 1,2,\dots,6 \end{array} \quad .$$

4.4.1 The System of Differential Equations

As shown in Figure 4.2, the model is composed of six internal, or endogenous, variables: GDP Per Capita, As Reported Earnings, Dividends, Buybacks, Retained Quarterly Earnings, and the S&P 500. These variables affect themselves and each other according to the formula depicted in (4.8). Additionally, the Federal Funds Rate is considered an exogenous variable, or forcing function. This variable is not influenced by the other model components, but it does affect all the others at each point in time according to the scalar d.

Because of the interconnected nature of the networked model, the effects of each variable on the six endogenous variables are additive for any given time t. Taking (4.8) and expanding it to show this additive effect, including the effect of the exogenous variable z_t, produces the derivative for each of the variables, $i = 1,2,\dots,6$, shown in equation (4.9).

$$\text{Change in variable } i = \sum_{j=1}^{6}\left[a_j \left(\frac{x_t^{(j)}}{b_j} - 80 \right) \right] + d_i z_t \tag{4.9}$$

Thus, by adding together the states of all the model variables at time t, the change in variable i is calculated from time t to $t+1$. This is the differential equation for variable i of the model (Saie, 2012:47).

Section 4.2.2 explained how each component of the model is abbreviated using the capital letters shown in the parentheses next to its name in Figure 4.2. Substituting the $x_t^{(j)}$ and z_t terms in (4.9) with the model abbreviations and listing each of the model components produces a system of six differential equations, each containing six terms plus the forcing function, given in (4.10). The notational form, $\dot{G}_t$, is the same as that used in the predator-prey model, where the dot over the component abbreviation denotes the change in the variable state at time t, the derivative, or the differential equation (Saie, 2012:50).

$$\begin{aligned}
\dot{G}_t &= a_{11}\left(\frac{G_t}{b_{11}}-80\right)+a_{12}\left(\frac{E_t}{b_{12}}-80\right)+a_{13}\left(\frac{D_t}{b_{13}}-80\right)+a_{14}\left(\frac{B_t}{b_{14}}-80\right)+a_{15}\left(\frac{R_t}{b_{15}}-80\right)+a_{16}\left(\frac{S_t}{b_{16}}-80\right)+d_1F_t\\
\dot{E}_t &= a_{21}\left(\frac{G_t}{b_{21}}-80\right)+a_{22}\left(\frac{E_t}{b_{22}}-80\right)+a_{23}\left(\frac{D_t}{b_{23}}-80\right)+a_{24}\left(\frac{B_t}{b_{24}}-80\right)+a_{25}\left(\frac{R_t}{b_{25}}-80\right)+a_{26}\left(\frac{S_t}{b_{26}}-80\right)+d_2F_t\\
\dot{D}_t &= a_{31}\left(\frac{G_t}{b_{31}}-80\right)+a_{32}\left(\frac{E_t}{b_{32}}-80\right)+a_{33}\left(\frac{D_t}{b_{33}}-80\right)+a_{34}\left(\frac{B_t}{b_{34}}-80\right)+a_{35}\left(\frac{R_t}{b_{35}}-80\right)+a_{36}\left(\frac{S_t}{b_{36}}-80\right)+d_3F_t\\
\dot{B}_t &= a_{41}\left(\frac{G_t}{b_{41}}-80\right)+a_{42}\left(\frac{E_t}{b_{42}}-80\right)+a_{43}\left(\frac{D_t}{b_{43}}-80\right)+a_{44}\left(\frac{B_t}{b_{44}}-80\right)+a_{45}\left(\frac{R_t}{b_{45}}-80\right)+a_{46}\left(\frac{S_t}{b_{46}}-80\right)+d_4F_t\\
\dot{R}_t &= a_{51}\left(\frac{G_t}{b_{51}}-80\right)+a_{52}\left(\frac{E_t}{b_{52}}-80\right)+a_{53}\left(\frac{D_t}{b_{53}}-80\right)+a_{54}\left(\frac{B_t}{b_{54}}-80\right)+a_{55}\left(\frac{R_t}{b_{55}}-80\right)+a_{56}\left(\frac{S_t}{b_{56}}-80\right)+d_5F_t\\
\dot{S}_t &= a_{61}\left(\frac{G_t}{b_{61}}-80\right)+a_{62}\left(\frac{E_t}{b_{62}}-80\right)+a_{63}\left(\frac{D_t}{b_{63}}-80\right)+a_{64}\left(\frac{B_t}{b_{64}}-80\right)+a_{65}\left(\frac{R_t}{b_{65}}-80\right)+a_{66}\left(\frac{S_t}{b_{66}}-80\right)+d_6F_t
\end{aligned} \tag{4.10}$$

Because every equation in the system includes inputs from all the economic variables in the model, the system describes the interrelatedness of the model's separate components and the effect of the federal funds rate F on each component. When the combined results of the individual differential equations are applied to the full system, the differential equations represent the rate of change of each variable per time step.

4.4.2 Euler's Forward Method

The interrelated behavior of the network components is inherent to the system of differential equations and captures the feedback that exists in the system as time progresses.

That is, the state of a variable at time t is directly related to its state at time $t-1$, as well as that of all the other variables.

In using the system of equations to replicate the real-world operations of the stock market, this change over time is approximated by taking initial values of the six model components and adding to them the calculated changes in their values over the next time step. These changes are given by the system differential equation, after the coefficients have been determined. This method is known as the Euler Forward Method, and it is formalized in equation (4.11) where t denotes the current time period, h indicates the step size, and $\dot{y}_t$ represents the predicted change at time t given the inputs to the model (Saie, 2012:69; Weisstein, 2015a).

$$y_{t+1} = y_t + h\dot{y}_t,\ t = 1, 2, \ldots \tag{4.11}$$

Expanding this representation to illustrate the values being calculated in the full economic model produces the set of equations in (4.12) where the notational form $\dot{G}_t$ represents the differential equation for each variable and h has a value of 1. Thus, by solving the differential equations and inserting an initial value, the value of each variable can be determined at any particular period as time progresses.

$$\begin{aligned} G_{t+1} &= G_t + h\dot{G}_t \\ E_{t+1} &= E_t + h\dot{E}_t \\ D_{t+1} &= D_t + h\dot{D}_t \\ B_{t+1} &= B_t + h\dot{B}_t \\ R_{t+1} &= R_t + h\dot{R}_t \\ S_{t+1} &= S_t + h\dot{S}_t \end{aligned} \tag{4.12}$$

The power of Euler's method lies in its predictive ability. The set of equations in (4.12) are used to fit the model to the data in order to determine the values of the coefficients in (4.10). When this is done, the system of differential equations is updated at each time t with actual data values, with the intent of matching the variable values in (4.12), that is G_{t+1}, E_{t+1}, etc., as closely as possible to the real data values at $t+1$.

After the coefficients in the system of differential equations are found, then Euler's method can be used to accurately predict the variable states at any point in the future given a single starting point. Alternatively, it can be used to estimate hypothetical states of the model, or system states that might have been achieved if different forcing function values had been input to the model. Thus, Euler's method is highly valuable to the analysis of dynamical systems and assists in characterizing the behavior of complex systems.

4.4.3 The System of Differential Equations in Matrix Notation

The model is simplified somewhat by defining the coefficients in the system of equations as matrices, or $\left[A_{ij}\right]_{6\times6}$, $\left[B_{ij}\right]_{6\times6}$, and $\left[D_i\right]_{6\times1}$, where $\left[B_{ij}\right]_{6\times6}$ is a matrix of the inverse of the b coefficients (Saie, 2012:50-52).

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} \quad B=\begin{pmatrix} \frac{1}{b_{11}} & \frac{1}{b_{12}} & \frac{1}{b_{13}} & \frac{1}{b_{14}} & \frac{1}{b_{15}} & \frac{1}{b_{16}} \\ \frac{1}{b_{21}} & \frac{1}{b_{22}} & \frac{1}{b_{23}} & \frac{1}{b_{24}} & \frac{1}{b_{25}} & \frac{1}{b_{26}} \\ \frac{1}{b_{31}} & \frac{1}{b_{32}} & \frac{1}{b_{33}} & \frac{1}{b_{34}} & \frac{1}{b_{35}} & \frac{1}{b_{36}} \\ \frac{1}{b_{41}} & \frac{1}{b_{42}} & \frac{1}{b_{43}} & \frac{1}{b_{54}} & \frac{1}{b_{45}} & \frac{1}{b_{46}} \\ \frac{1}{b_{51}} & \frac{1}{b_{52}} & \frac{1}{b_{53}} & \frac{1}{b_{54}} & \frac{1}{b_{55}} & \frac{1}{b_{56}} \\ \frac{1}{b_{61}} & \frac{1}{b_{62}} & \frac{1}{b_{63}} & \frac{1}{b_{64}} & \frac{1}{b_{65}} & \frac{1}{b_{66}} \end{pmatrix} \quad D=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix}$$

Likewise, the economic variables in the model make up the $\left[X\right]_{6\times1}$ matrix, while the federal funds rate composes the $\left[F\right]_{1\times1}$ matrix.

$$X=\begin{pmatrix} G \\ E \\ D \\ B \\ R \\ S \end{pmatrix}$$

$$F=\left(F\right)$$

Using these matrices, the functional form of the system of differential equations is also simplified. First, two vectors of 1's are introduced, $\left[1\right]_{1\times6}$ and $\left[1\right]_{6x1}$, and the new, compact version of the differential equations is given in matrix notation. This is depicted in equation (4.13).

$$\dot{X}_t=\left(\left(A\circ\left(\left(X_t\left[1\right]_{1x6}\right)\circ B\right)\right)-80A\right)\left[1\right]_{6x1}+DF \tag{4.13}$$

Equation (4.13) uses two Hadamard products which are defined as the element-wise multiplication of two matrices, G and H, where $[G \circ H]_{ij} = [G]_{ij}[H]_{ij}$, for all $1 \le i \le m, 1 \le j \le n$ and $G, H, G \circ H \in R^{mxn}$ (Million, 2007; Saie, 2012:51-52). The resulting matrix, $\dot{X}_t$, is a 6×1 vector that gives the rate of change of the model at time t, and is input to equation (4.12) to get the next set of state variables at time $t+1$.

$$\dot{X}_t = \begin{pmatrix} \dot{G}_t \\ \dot{E}_t \\ \dot{D}_t \\ \dot{B}_t \\ \dot{R}_t \\ \dot{S}_t \end{pmatrix}$$

4.5 Solving for the Coefficients of the Differential Equations

To solve the inverse problem and find the coefficients of the system of differential equations, least squares nonlinear minimization is applied using the mathematical program shown in equations (4.14) through (4.19).

$$\text{Minimize} \quad f(x) = \sum_{i=1}^{n}\sum_{t=1}^{T}\frac{t}{T}\left(y_t^{(i)} - \hat{y}_t^{(i)}\right)^2 \tag{4.14}$$

$$\text{subject to} \quad b_{ij} \le 1 \qquad \text{for } i, j = 1, 2, ..., 6 \tag{4.15}$$

$$b_{ij} \ge 0.000001 \qquad \text{for } i, j = 1, 2, ..., 6 \tag{4.16}$$

$$d_{ij} \le 0 \qquad \text{for } i = 2, 3, 4 \tag{4.17}$$

$$a_{ij}, b_{ij}, d_i \in R \qquad \text{for } i, j = 1, 2, ..., 6 \tag{4.18}$$

$$t \in Z^+ \tag{4.19}$$

In this program, the objective function minimizes the total sum of the squared residuals, across all values of t and all six model components, by choosing coefficient values of a_{ij}, b_{ij}, and d_i for the system of differential equations so that the difference between the predicted value of variable i, at each point t, denoted as $\hat{y}_t^{(i)}$, and its actual value, $y_t^{(i)}$, is as close to 0 as possible. This is accomplished using the generalized reduced gradient (GRG) algorithm, which seeks to find the lowest local minima by changing the coefficients of the system in a direction that results in the greatest reduction of the sum of squared errors (SSE) (Saie, 2012:52-53).

In the case of the three approaches used to fit the data discussed in Section 4.3.3, the values of $\hat{y}_t^{(i)}$ and $y_t^{(i)}$ take on different meanings. Under the first approach, in which the predicted data points are made to match the actual data values, $\hat{y}_t^{(i)}$ represents the predicted datum for component i at time t and $y_t^{(i)}$ is the actual value. The second approach attempts to match the values of the differential equations to the slopes between adjacent actual data points. Hence, in this case, $\hat{y}_t^{(i)}$ symbolizes the value of the differential equation for component i at time t, and $y_t^{(i)}$ is the value of the vertical change in component i from time $(t-1)$ to time t. Similarly for the third approach, $\hat{y}_t^{(i)}$ represents the secant for the predicted points at times $(t-\nu)$ to $(t+\nu)$, where ν is the step size, and $y_t^{(i)}$ is the secant of the actual data values over the same time period. When the sum of squared differences between $\hat{y}_t^{(i)}$ and $y_t^{(i)}$ are calculated using the three separate approaches, different fits are obtained because different objectives are being pursued. The first approach simply tries to match the predicted points to the actual data, whereas the objective of the other two approaches is to match the slopes or average slopes in order to better replicate the trends of the actual data.

The objective function in the nonlinear program is further influenced by the weighting applied to each squared residual. In order to better predict model behavior over the validation set, the weight, $\frac{t}{T}$, forces the model to choose coefficients that will weigh more heavily at the end of the training period. In other words, system behavior early in the time-ordered data set will have a lesser impact on SSE than behavior that occurs late in the data set. If the training data is set at 55 quarters, with the remaining 14 quarters used for validation, then setting $T = 55$ forces the minimization program to weight the last quarter of the training data most, thereby allowing it the greatest impact on coefficient selection, which also affects the accuracy of the model over the validation period immediately succeeding the training period.

The first and second constraints, equations (4.15) and (4.16), require that all entries in the B matrix be less than or equal to one, but greater than 0. This causes the b_{ij} coefficients to act as tipping points in the system, as discussed in Section 4.4.1. Since the software used for the GRG algorithm will not take "hard" inequality constraints such as $b_{ij} > 0$, this constraint is approximated in the nonlinear program with a value close to zero, or $b_{ij} \geq 0.000001$, as shown in (4.16). The third constraint, equation (4.17), dictates that the D coefficients for As Reported Earnings, Dividends, and Buybacks be less than or equal zero, meaning they have an inverse relationship to the interest rate value F. Equation (4.18) constrains all coefficients to be real numbers, and equation (4.19) simply requires that the time variable, t, be composed of positive integer values.

As noted in Chapter III, any solution found by the program will not be unique, and similar predicted values can be obtained by using different values of a_{ij}, b_{ij}, and d_i. Furthermore, any solution obtained is specific only to the set of data input to the program. As noted by Helmbold, the parameters obtained through the nonlinear optimization are "particular constants," unique only to the problem context and set of data employed to solve for them. They are not universal and cannot be used with other data. If new data is applied to the system, new coefficients must also be found that are particular to that problem and that data (Helmbold, 1994; Saie, 2012:53).

4.5.1 The Method of Least Squares

Solving the mathematical program of equations (4.14) through (4.19) requires using the method of least squares. As discussed in Chapter III, the method of least squares is a procedure for finding the curve that best fits a given set of data points by attempting to minimize the sum of the errors between each actual data point and its corresponding predicted point. This is commonly done by finding the vertical distance from an actual data point to the data point predicted by a model, and squaring the distance. Summing the squared differences over all the data points on a fitted curve yields the sum of squared errors (SSE) for the model. Linear and nonlinear programming can then be used to minimize the SSE as far as possible by choosing model coefficients that move the predicted data points closer to the actual data points. Choosing these parameters in the context of this economic model solves the inverse problem and facilitates further analysis on the model.

The formula for finding the vertical error between the actual and predicted points is defined by taking the difference of the actual data point and its predicted point at each point in time, as shown in equation (4.20).

$$e_t^{(i)} = \left(y_t^{(i)} - \hat{y}_t^{(i)}\right), i = 1, 2, \ldots, 6 \tag{4.20}$$

By squaring this difference, the resulting residual will always be positive, and the sum of the squared errors is found using equation (4.21):

$$SSE = \sum_{t=1}^{T} \left(y_t^{(i)} - \hat{y}_t^{(i)} \right)^2, \, i = 1, 2, \ldots, 6, \tag{4.21}$$

where the predicted value $\hat{y}_t^{(i)}$ is the result of equation (4.12) for each of the six model components, $i = 1, 2, \ldots, 6$, at each time step t.

The overall utility, or fit, of the model is assessed using the coefficient of determination, or R-squared value (Bowerman, O'Connell, & Koehler, 2005:114-116). This quantity makes use of the proportion of variation that is explained by the model. By finding the mean of each set of economic data, denoted as $\overline{y}^{(i)}$, the total variation in the data is first found. This is done by taking the sum of the squared distances of the actual data from its mean, as shown in equation (4.22).

$$\text{Total Variation in Data} = \sum_{t=1}^{T} \left(y_t^{(i)} - \overline{y}^{(i)} \right)^2, \, i = 1, 2, \ldots, 6 \tag{4.22}$$

Then, the variation explained by the model is found using equation (4.23).

$$\text{Variation Explained by Model} = \sum_{t=1}^{T} \left(\hat{y}_t^{(i)} - \overline{y}^{(i)} \right)^2, \, i = 1, 2, \ldots, 6 \tag{4.23}$$

When these two values are combined into the R-squared value, they form a ratio of the amount of variation explained by the model for each of the six model components, as shown in (4.24).

$$R_i^2 = \frac{\sum_{t=1}^{T} \left(\hat{y}_t^{(i)} - \overline{y}^{(i)} \right)^2}{\sum_{t=1}^{T} \left(y_t^{(i)} - \overline{y}^{(i)} \right)^2}, \, i = 1, 2, \ldots, 6 \tag{4.24}$$

The R-squared measure provides a convenient way of describing the model's fit. Similarly, the SSE is modified to show the average error in the model, or the average variation per data point that is unexplained by the model. This is found by computing the mean squared error (MSE), or the average of squared errors across an entire fitted curve. MSE is calculated by dividing the SSE from (4.21) by the number of observations in the data. Equation (4.25) shows this formula. This measure is used to make an equal comparison between the model fit over the training data and the validation data.

$$MSE^{(i)} = \frac{SSE^{(i)}}{T} = \frac{\sum_{t=1}^{T}\left(y_t^{(i)} - \hat{y}_t^{(i)}\right)^2}{T}, i = 1, 2, \ldots, 6 \tag{4.25}$$

One side effect of the method of least squares is that it tends to weight large residuals heavier than small residuals. Where the model has large departures from the actual data, the square of this distance accentuates its effect on the SSE. The method of least squares in turn tries to reduce these large errors first because of their greater magnitude and consequently heavier weighting they receive in the SSE equation. As a result, when coefficients are being determined via nonlinear programming, the coefficients are chosen to reduce the largest errors in the model first. As discussed in Section 3.4, this has the effect of finding a localized minimum on the problem's polytope surface, possibly in lieu of the global minimum. Minimizing the absolute values of the residuals would not cause this to happen, but the sum of the squared residuals is used here because they can be treated as a continuous differentiable fit (Weisstein, 2015b).

4.6 Analyzing the Results

Besides the measures of model fit described in Section 4.5.1, the model is analyzed via several other methods. This section will explain the methodology used to evaluate the model residuals, calculate the Euler curves, and conduct hypothesis testing on the hypothetical scenarios. The methodology for finding the prediction intervals of a particular point in the model will also be presented.

4.6.1 Residual Analysis

Residual analysis is performed to evaluate the adequacy and fit of the model. In the method of least squares, a well-fitting model should satisfy four criteria in regards to its residuals.

1. For any given combination of values, the population of residuals should have a mean of zero.
2. The residuals over the course of the model should have a constant variance.
3. The population of residuals should be normally distributed.
4. Each residual should be statistically independent of all other residuals in the population (Bowerman *et al*., 2005:145).

The first criterion is verified by calculating the means of each set of residuals, and by visually inspecting a plot of the residuals distributed over t. The same plot of the residuals can be used to verify the second criterion. If the residuals are randomly distributed across the plot with no discernible pattern, the residuals are considered to have constant variance. This method will also verify the fourth criterion, although this assumption is often violated by time series data that is autocorrelated. The third criterion is verified by visually examining the residuals on a normal probability plot. If the residuals are evenly distributed along the forty-five degree line of this plot, the residuals are assumed to be normally distributed and "well-behaved."

The third criterion is also evaluated using a measure of the model's goodness-of-fit. The Shapiro-Wilk test is used to produce a quantitative metric of residual normality. This statistic is computed using equations (4.26) and (4.27), where $e_{(k)}$ is the kth smallest residual, $\overline{e}$ is the mean of the residual values, m is a vector of the expected values of the order statistics of independent and identically distributed random variables, and V is the covariance matrix of the order statistics (Shapiro & Wilk, 1965).

$$W^{(i)} = \frac{\left(\sum_{k=1}^{T} a_k e_{(k)}^{(i)}\right)^2}{\sum_{k=1}^{T}\left(e_k^{(i)} - \overline{e}^{(i)}\right)^2}, i = 1, 2, ..., 6 \tag{4.26}$$

$$\left(a_1, ..., a_T\right) = \frac{m'\mathrm{V}^{-1}}{\sqrt{\left(m'\mathrm{V}^{-1}\mathrm{V}^{-1}m\right)}} \tag{4.27}$$

The $W^{(i)}$ statistic in equation (4.26) is then used in a hypothesis test as shown in (4.28). If the statistic is below the predetermined critical values outlined by Shapiro and Wilk at a given level of α, the null hypothesis is rejected, and the distribution of the residuals is considered non-normal (Shapiro & Wilk, 1965).

$$\begin{matrix} H_0 : W^{(i)} \geq W_{Crit, \alpha=0.05} \\ H_1 : W^{(i)} < W_{Crit, \alpha=0.05} \end{matrix}, i = 1, 2, ..., 6 \tag{4.28}$$

The hypothesis test is also done using calculated *p*-values, where a *p*-value less than a certain threshold, which is $\alpha = 0.05$ for this research, results in rejecting the hypothesis that the residuals are normal.

While the Shapiro-Wilk test provides valuable insight into the adequacy of the model, it is known to be a conservative estimate of normality. If a set of data passes the Shapiro-Wilk test and satisfies the other assumptions addressed in residual analysis, the model is considered adequate. When it does not pass the Shapiro-Wilk test, however, this is only one indication of non-normality but does not invalidate the adequacy of the model. In fact, slight deviations from normality in the normal probability plots are usually acceptable, as most statistical inference tests are known to be robust to deviations from normality (Douglas C. Montgomery Elizabeth A. Peck & Vining, 2012:136).

Vining explains that using the raw residuals from the model, or those calculated using equation (4.20), results in a set of residuals with non-constant variance. That is, the variance of the residuals depends on the distance of residuals from the centroid of the data (Vining, 2011:106). This issue is commonly resolved by using the externally studentized residuals, which is the best standardization available (Vining, 2011:106). These are calculated by first finding the Mahalanobis distance of each data point, denoted as h_{tt}. These distances are equal to the diagonal elements of the "hat" matrix, computed using equation (4.29). The X matrix referenced here is the vector of time periods, $t_1 = 1, t_2 = 2, ...$, used for the model training data.

$$H = X(X'X)^{-1}X' \tag{4.29}$$

After the h_{tt} distances are obtained, these values are used to "externally studentize" the raw residuals using equation (4.30).

$$\tau_t^{(i)} = \frac{e_t^{(i)}}{\sqrt{MSE_{-t}^{(i)}(1-h_{tt})}}, i = 1, 2, ..., 6 \tag{4.30}$$

This equation standardizes each raw residual at time t for model component i, by dividing the residual by the square root of the mean square error multiplied by the quantity of one minus its hat distance. The mean square error is calculated using all the residuals for component i except the residual for the time period in question at time t. Standardizing the residuals in this manner has the desirable effect of keeping the numerator statistically independent from the denominator, since the MSE in the denominator was calculated with the residual in the numerator excluded.

This also gives the resulting set of studentized residuals the convenient property of being *t*-distributed, which makes the identification of outliers readily apparent. If any single residual is disproportionately large compared to the other studentized residuals, it is assumed to lie in the tails of the distribution as an outlier. As such, it may be considered for removal from the model in order to improve the model fit (Vining, 2011:106).

4.6.2 Calculating the Euler Curves

Section 4.4.2 detailed the methodology of the Euler Forward Method as applied to the model. It described how Euler's numerical equation, repeated here in equation (4.31), is applied to the six model components, equation set (4.32), in order to calculate new system states as time progresses.

$$y_{t+1} = y_t + h\dot{y}_t,\ t = 1, 2, \ldots \tag{4.31}$$

$$\begin{aligned} G_{t+1} &= G_t + h\dot{G}_t \\ E_{t+1} &= E_t + h\dot{E}_t \\ D_{t+1} &= D_t + h\dot{D}_t \\ B_{t+1} &= B_t + h\dot{B}_t \\ R_{t+1} &= R_t + h\dot{R}_t \\ S_{t+1} &= S_t + h\dot{S}_t \end{aligned} \tag{4.32}$$

This method is now used to calculate the predicted performance of the data over T time periods, beginning with a single vector of inputs at $t = 1$. This vector,

$$X_1 = \begin{pmatrix} G_1 \\ E_1 \\ D_1 \\ B_1 \\ R_1 \\ S_1 \end{pmatrix},$$

plus the value of the federal funds rate, $F_1 = (F_1)$, is entered into the system of differential equations shown in equation set (4.10) to get the vector

$$\hat{\dot{X}}_1 = \begin{pmatrix} \hat{\dot{G}}_1 \\ \hat{\dot{E}}_1 \\ \hat{\dot{D}}_1 \\ \hat{\dot{B}}_1 \\ \hat{\dot{R}}_1 \\ \hat{\dot{S}}_1 \end{pmatrix}.$$

Vectors X_1 and $\hat{\dot{X}}_1$ are then used in equations (4.32) to calculate the new system state at $t = 2$, or $\hat{X}_2$, which, with F_2, is applied to the system of differential equations and (4.32) to obtain $\hat{X}_3$, and so forth. Plotting the resulting set of predicted system states as t increases illustrates the dynamic nature of the system. It also permits analysis into the relationships of the model components, based on the coefficients of the system of differential equations, and allows predictions of future system states without updating the model with actual data (Saie, 2012:52).

The fit of the Euler curves is assessed using the same techniques outlined in Sections 4.5.1 and 4.6.1. If the Euler curves adequately replicate the behavior of the actual data, they are used as a baseline to compare against a new set of hypothetical scenario curves. The hypothetical curves are calculated using the same Euler methodology, only the forcing function values, F, are altered to simulate alternative scenarios. This enables a sensitivity analysis of the federal funds rate and illustrates what might have happened if the Federal Reserve had implemented different interest rates than what appears in the actual data.

4.6.3 Analyzing the Hypothetical Scenarios

After Euler curves are calculated for the hypothetical scenarios, analysis is performed to determine whether the new federal funds rate caused a significant change in system behavior. Hypothesis testing compares the means of the alternative system states against the baseline curves, and prediction intervals assess whether the end state of the alternative system is different than the baseline end state.

The null hypothesis states that there is no change between the means of the baseline set of Euler curves and the means of the hypothetical scenarios. The alternative hypothesis states the opposite, that there is a significant difference between the two sets of means. These two hypotheses are shown in notational form in equation (4.33).

$$\begin{aligned} H_0 &: \mu_{Baseline}^{(i)} = \mu_{Hypothetical}^{(i)} \\ H_1 &: \mu_{Baseline}^{(i)} \neq \mu_{Hypothetical}^{(i)} \end{aligned} \tag{4.33}$$

The hypotheses are tested using the *t*-statistic shown in equation (4.34), where $\sigma_{Baseline}^{(i)}$ is the standard deviation of the baseline data and n is the number data points.

$$t = \frac{\mu_{Baseline}^{(i)} - \mu_{Hypothetical}^{(i)}}{\sigma_{Baseline}^{(i)} / \sqrt{n}} \tag{4.34}$$

The null hypothesis is rejected if the *p*-value associated with the *t*-statistic is less than the desired α value of 0.05. If the null hypothesis is rejected, this is significant evidence that the hypothetical scenario is different than the baseline. A *p*-value greater than 0.05 fails to reject the null hypothesis and leads to the conclusion that the two cases are not statistically significantly different.

The maximum errors in the curves and the end states of the hypothetical scenario and the baseline case are compared using prediction intervals. As opposed to the confidence interval, which calculates a $100(1-\alpha)\%$ region of confidence around a mean, a prediction interval provides an $100(1-\alpha)\%$ interval around an individual data point. For each predicted data point, the intervals indicate with $100(1-\alpha)\%$ certainty that the true data value lies within the range specified. The formula for computing a prediction interval is shown in equation (4.35), where $\hat{x}^{(i)}_{n_{Hypothetical}}$ is the predicted value for model component i at time $t = n$, the last time period modeled. Of the other parameters in the model, $t_{n,\alpha/2}$ is the *t*-distributed critical value, $s^{(i)}$ is the standard error of the data from its mean, and the formula for the distance value is given in equations (4.37) and (4.38) (Bowerman *et al.*, 2005:108-114).

$$\left[\hat{x}^{(i)}_{n_{Hypothetical}} \pm t_{n,\alpha/2} s^{(i)} \sqrt{1 + \text{Distance value}}\right] \tag{4.35}$$

$$s^{(i)} = \sqrt{\frac{SSE^{(i)}}{n}} \tag{4.36}$$

$$\text{Distance value} = \frac{1}{n} + \frac{\left(t - \bar{t}\right)^2}{SS_{tt}} \tag{4.37}$$

$$SS_{tt} = \sum_{t=1}^{n} t^2 - \frac{\left(\sum_{t=1}^{n} t\right)^2}{n} \tag{4.38}$$

Prediction intervals are valuable tools for assessing the behavior between two data curves. If a change in the federal funds rate causes a substantial shift in the model dynamics over time, this is reason for concluding that the federal funds rate, as a forcing function, is a powerful influence on the system, capable of inflicting drastically different long-term results.

4.7 Summary

This chapter described the data, model, and methodology used to understand the dynamics that exist between the six economic variables. First, it described the manner in which economic data is collected and processed. It then introduced the model and its system of differential equations, and it explained the method for determining the coefficients of the equations via nonlinear programming and the method of least squares. It concluded by discussing the methodology used to analyze the model, and provided statistical tests for detecting significant differences between a baseline case and alternative scenario.

An advantage to using a system of differential equations is its ability to capture the interrelatedness of the variables of the system. The magnitude of each coefficient illustrates the influence that each variable exerts on itself and the others. In particular, the weight assigned to the exogenous variable shows how the forcing function affects the variables individually and collectively, allowing for in-depth sensitivity analysis when the model is implemented and the results compiled.

While the data input to the model is correct, the true sources of variability in a system as large and as complex as the U.S. stock market cannot be known with certainty and can only be conjectured. Not all direct and indirect relationships can be captured, and not all influential variables can be included. Many influences on the market and the economy are largely qualitative and are thus difficult to measure and include in a quantitative model. These influences may include congressional or state legislation, whether intentionally directed at the economy and its components or not, or simply the behavioral and psychological dynamics of investors. Therefore, this model assumes that such influences are found largely in the variability of the data, while the primary elements of variance are captured by the model's variables.

The implementation of the model and its results will be detailed in Chapter V. The methods from Chapter IV will be further developed as necessary for implementation, and analysis of the results and validation will be discussed.

V. Implementation and Analysis

5.1 Chapter Overview

The methodology from Chapter IV was implemented by constructing a system of differential equations for the conjectured model, then fitting the model to the actual economic data via the method of least squares. Chapter V will detail this process and discuss the results of the model and its prediction accuracy. The chapter will also summarize three "what-if" scenarios to simulate how the economy and the stock market may have reacted to hypothetical actions taken by the Federal Reserve on the federal funds rate during and after the 2008 recession.

5.2 Data Collection and Model Formulation

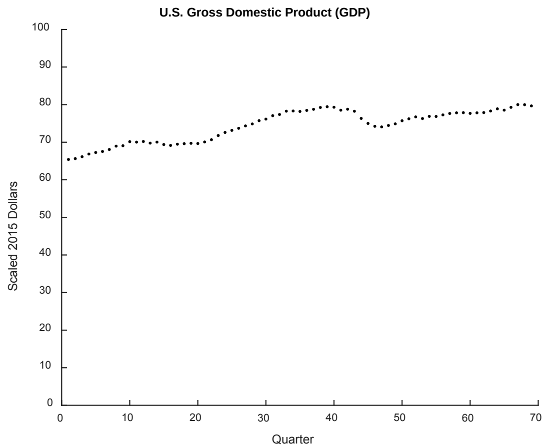
Data was collected on the six variables of interest and the forcing function for the period of January 1998 to March 2015. This particular time period was chosen as it is the most relevant to ongoing economic circumstances, especially considering the volatility currently present in stock markets; and because stock buyback and dividend data for the S&P 500, which are central pieces of the conjectured model, was calculated differently prior to 1998 (H. Silverblatt, personal communication, 21 September 2015). Furthermore, the end date of 1st quarter 2015 was selected because it was the last period for which complete data sets were available when the study commenced during the summer of 2015.

All the data was preprocessed according to the methodology outlined in Chapter IV, with monetary values being adjusted for inflation using the CPI and set equal to April 2015 dollars. The seven variables were then scaled by multiplying each data value by a scalar, listed in Table 5.1. This resulted in each dataset having a maximum value equal to 80 and making the variables directly comparable to one another, without a single variable receiving more weight during model fitting due to the disproportionate magnitude of its values. The complete set of adjusted and unadjusted data is included in Appendix A.

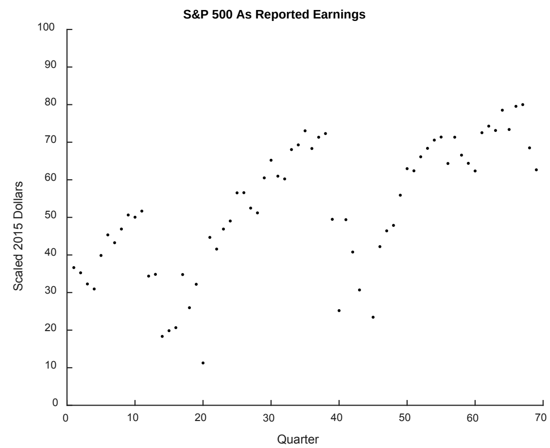
Table 5.1: Variable data weights

Economic Variable	Scalar
GDP Per Capita	0.001438255
As Reported Earnings	0.323297518
Dividends	0.851203541
Buybacks	0.406304345
Retained Quarterly Earnings	1.127028231
S&P 500	0.039323295
Federal Funds Rate	4.496908375

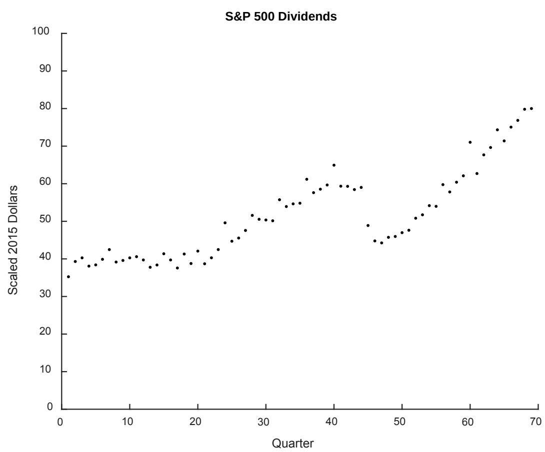
Plots of the scaled data show the change in the variables over the time period of interest, and some interaction or correlation is apparent in their movement, as depicted in Figure 5.1. Also obvious in the plots are several significant fluctuations. The first observable deviation from the average trend is seen in the As Reported Earnings and S&P 500 plots around the 10th quarter. This spike in the data represents the "dot com" boom of the late 1990's. Following this period is a dip in several of the plots representing the recession of 2001, as mentioned in Chapters I and III. The change in the Federal Funds rates for these two periods is apparent, as increases in the rate are observed around the 10th quarter followed by a decrease around the 15th quarter when interest rates were dropped to bring the U.S. economy out of the recession. This drop in interest rates is then followed by sharp increases in all six plots, as earnings, dividends, buybacks, and stocks all benefit from the low interest rates and the housing boom of 2006. This boom is followed by the crash of 2007-2008, when all the plots drop quickly, and the earnings plots register extreme outliers in the negative. The Federal Reserve again drops interest rates around this point, and the variables gradually recover to the levels seen at the end of each plot.



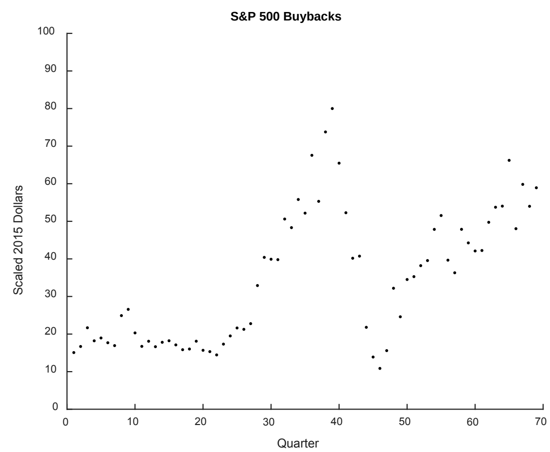
(a) GDP Per Capita



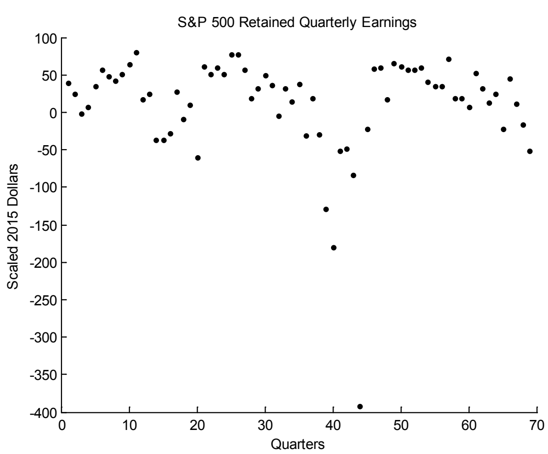
(b) As Reported Earnings



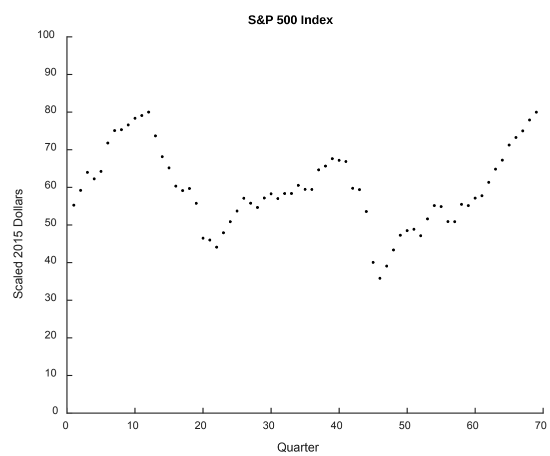
(c) Dividends



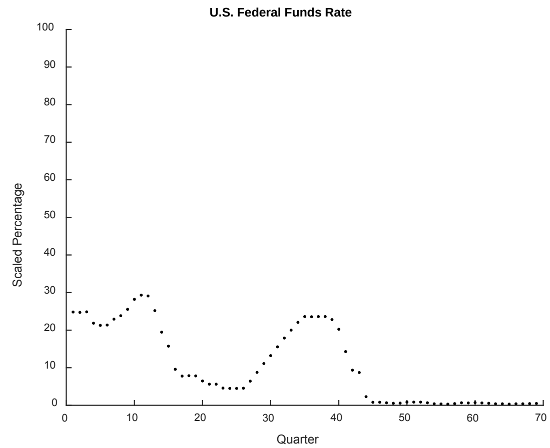
(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500



(g) Federal Funds Rate

Figure 5.1: Plots of scaled model data

The slopes and secants for each of the six variables were calculated according to the procedure in Section 4.3.2. The slopes were derived over single, consecutive quarters by taking the value of a variable for a particular quarter and subtracting from it the value of the same variable from the previous quarter, thus recording the change between the two time periods, as follows in equation (5.1):

$$m_t^{(i)} = \frac{y_t^{(i)} - y_{t-1}^{(i)}}{t-(t-1)}, \text{ for all } i,\ t = 2,\ldots,69 \qquad (5.1)$$

The secants of each data series were calculated in like manner, depicted in equation (5.2), except the slope at each point was a function of the series values just prior to and after the current time period:

$$c_t^{(i)} = \frac{y_{t+1}^{(i)} - y_{t-1}^{(i)}}{(t+1)-(t-1)}, \text{ for all } i,\ t = 2,\ldots,69 \qquad (5.2)$$

The data was then divided into two groups: a "training" set consisting of 80% of the data, or rather the data representing the period of January 1998 to September 2011 (55 quarters);

and a “validation” set consisting of 20% of the data, or the period from October 2011 to March 2015 (14 quarters).

5.3 Fitting the Model to the Data

With the scaled values, slopes, and secants in place, the differential equations from (4.12) were converted to their computational form; and when the values from the economic variables and the federal funds rate for a particular quarter were used as inputs to the differential equations, the value for each variable was predicted for the succeeding quarter. The following equations, as introduced in (4.12) and repeated here as (5.3), are now implemented in the model for every value of t, in single time steps $(h=1)$, from January 1998 to September 2011.

$$\begin{aligned} G_{t+1} &= G_t + \dot{G}_t \\ E_{t+1} &= E_t + \dot{E}_t \\ D_{t+1} &= D_t + \dot{D}_t \\ B_{t+1} &= B_t + \dot{B}_t \\ R_{t+1} &= R_t + \dot{R}_t \\ S_{t+1} &= S_t + \dot{S}_t \quad . \end{aligned} \tag{5.3}$$

The sum of square errors is minimized by attempting to solve the nonlinear program in (4.14) – (4.19) using the GRG method. The coefficients for the system of differential equations introduced in Section 4.4.3, a_{ij}, b_{ij}, and d_i, are found so that the SSE is minimized and the value of $\hat{y}_t^{(i)}$ is as close to $y_t^{(i)}$ as possible, thus fitting the model-predicted data to the actual data.

5.3.1 Weighted Model

The coefficients for the system of differential equations were first found by minimizing the SSE for the scaled data values using the weighted formula described in equation (4.14).

These coefficients are shown in Table 5.2 through Table 5.4. The full coefficients as used in the model, extending to 15 digits, are given in Appendix B.

Table 5.2: *A*-coefficients in matrix form

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.00463	0.01746	0.00128	-0.01354	-0.00056	-0.00313
As Reported Earnings	0.06614	0.02354	-0.08988	-0.07495	-0.00436	-0.17161
Dividends	0.01488	0.03573	-0.06107	0.00562	0.00030	0.00388
Buybacks	0.05825	0.03662	-0.01656	-0.00882	0.01555	0.06732
Retained Quarterly Earnings	0.22324	-0.10349	-0.02159	-0.43421	-0.07988	-0.66327
S&P 500	-0.00982	0.06008	-0.02175	-0.03253	-0.00472	-0.09921

Table 5.3: *B*-coefficients in matrix form

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.50024	0.50009	0.50013	0.50082	0.50044	0.50261
As Reported Earnings	0.46769	0.49005	0.49326	0.57071	0.48745	0.52218
Dividends	0.49549	0.49237	0.50355	0.49509	0.50046	0.53872
Buybacks	0.62435	0.60451	0.48433	0.56389	0.61678	1.00000
Retained Quarterly Earnings	0.31782	0.66251	0.54343	0.51993	0.50626	0.32334
S&P 500	0.49341	0.45745	0.54159	0.42462	0.49586	0.48550

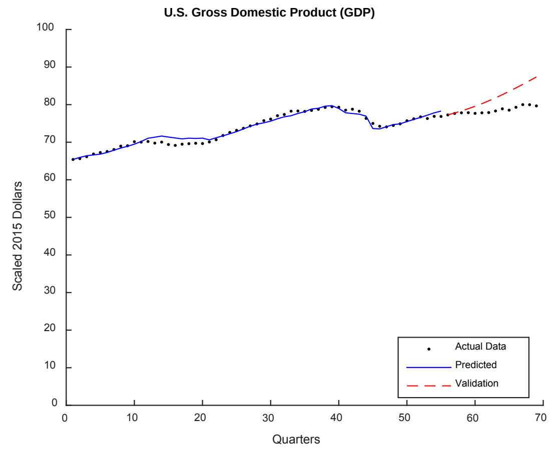
Table 5.4: *D*-coefficients in matrix form

	Federal Funds Rate
GDP Per Capita	0.01319
As Reported Earnings	0.00000
Dividends	-0.01422
Buybacks	0.06335
Retained Quarterly Earnings	1.33509
S&P 500	0.23105

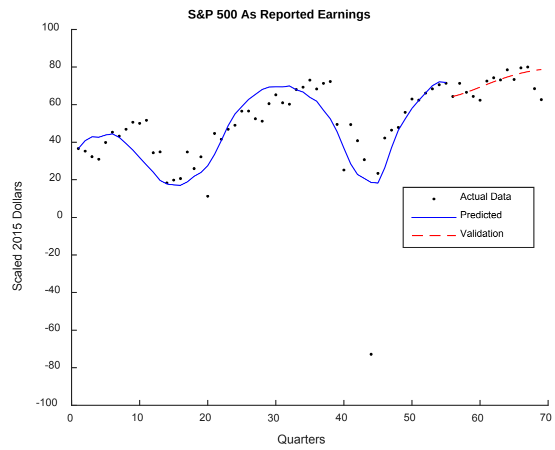
Then, the predicted values were calculated over the range of the validation set using Euler’s method and compared to the actual data. This was accomplished by applying the above-determined coefficients to the system of differential equations and using the predicted values for each of the six variables as the inputs for each of the variables over the next period as described in Section 4.4.2, thus producing an entirely predicted curve, independent of actual data, over the length of the validation period. Figure 5.2 shows the fit of the training data using the solid blue line, as well as that of the validation data predicted over the last 14 quarters, depicted by the dashed red line.

As seen in the plots, the model fits some of the variables well, while others are less precise, particularly across the validation period. The model curves are especially accurate in emulating the upward and downward trends of the data, with a few exceptions. Since the predictive potential of the model depends on its ability to capture the dynamic relationships of the model components, capturing these trends is even more important than simply matching the model to the actual data values. As time progresses, inputs can be injected to update the model, but it should be able to independently recognize critical tipping points in the data as discussed in Chapter IV. If it simulates these tipping points and downward or upward trends correctly, the model is more likely to predict future market surges and crashes, bull and bear markets. The plots in Figure 5.2 suggest it does this quite well.

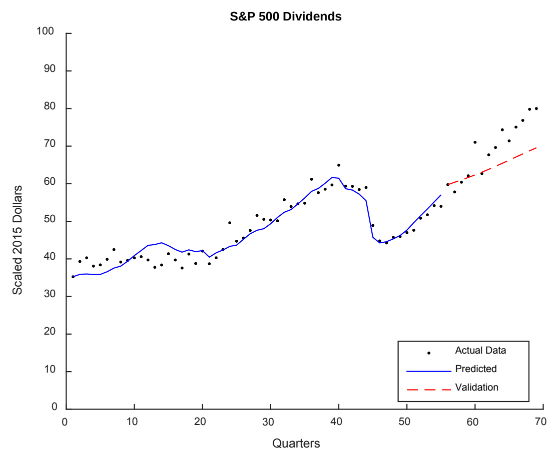
This subject will be discussed further in Section 5.7.1, but it is sufficient to state here that tipping points in the model can also be anticipated in advance through observation of two indicators: the first and second derivatives of the model component curves. As the first derivative, or the value of the differential equation for variable *i* approaches zero, the model itself approaches a tipping point. In similar fashion, as the second derivative grows smaller, the model approaches an inflection point at which the model's rate of change will begin to slow as it progresses toward the next tipping point.



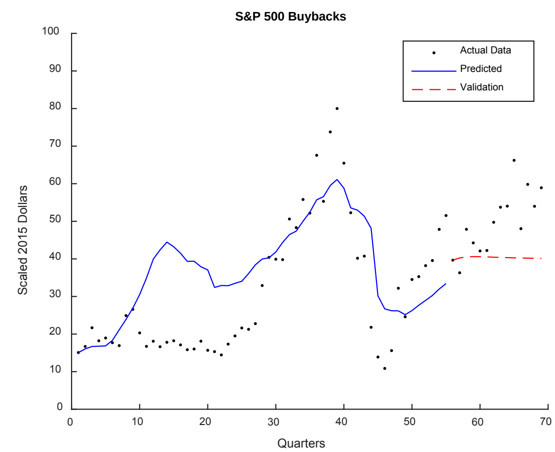
(a) GDP Per Capita



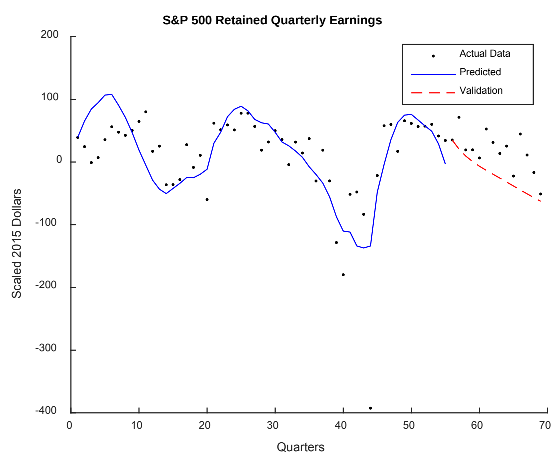
(b) As Reported Earnings



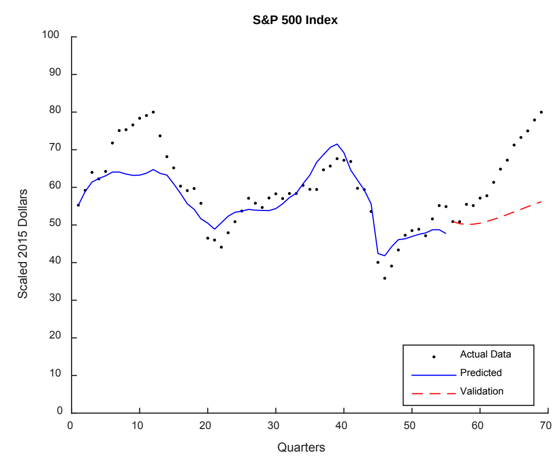
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.2: Plots showing fitted training data and validation data overlaying actual system data (black)

In addition to the trends, the model fits the data itself. The modeled GDP fits the data points well, while correctly predicting future behavior over the validation set, indicating the model and its determined coefficients are well-suited to that specific data set. In like manner, the model fits the dividend and buyback data relatively well. Departures from the actual data are obvious in the validation set however, indicating the model or its coefficients lack accuracy when operating independently of actual data inputs. The model is less well-fitted to the two sets of earnings data, but these data series are more volatile by nature, so less accuracy is expected. Most importantly however, is the good fit obtained on the S&P 500 data over the training set and the validation set. Overall, the model accurately predicts trends in the data across the training and validation sets even if the residual errors of the model are large at times.

A simple way of assessing the model fit is via its R-squared values, introduced in equations (4.22) - (4.24), and shown again here as equation (5.4).

$$R_i^2 = \frac{\sum_{t=1}^{T}\left(\hat{y}_t^{(i)} - \overline{y}^{(i)}\right)^2}{\sum_{t=1}^{T}\left(y_t^{(i)} - \overline{y}^{(i)}\right)^2}, i = 1, 2, \ldots, 6 \tag{5.4}$$

This formula examines the variation in the data as described by the model, compared to the total variation in the data away from its own mean. When these values are calculated for the predicted data, as shown in Table 5.10, the superior fits of GDP Per Capita and Dividends are shown by their values above 0.9. While the R-squared values for the other model components are lower, all of the values are above 0.5, which is noteworthy considering the volatile nature of the underlying data and economic systems in general.

Table 5.5: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	0.9028
As Reported Earnings	0.6528
Dividends	0.9386
Buybacks	0.6025
Retained Quarterly Earnings	0.7446
S&P 500	0.5505

The result of the model fits are also observed quantitatively by examining the unweighted mean square errors (MSE) that are generated by the weighted model. The unweighted residuals are used here to show the fit of the model over both the training and validation sets, without being biased by the weights in the training set, and it is calculated simply by using equation (4.25). The results for these errors are presented in Table 5.6, further illustrating which economic variables perform best in the model.

Table 5.6: Mean Square Errors Based on Data

	Training Set Mean Square Errors	Validation Set Mean Square Errors
GDP Per Capita	0.6692	16.2796
As Reported Earnings	252.0881	35.0323
Dividends	6.6599	43.0044
Buybacks	192.3542	157.5151
Retained Quarterly Earnings	2880.4066	2051.3376
S&P 500	33.9630	197.4595
Total	3366.1410	2500.6285

As already noted, the most volatile data series are those that involve quarterly earnings, namely As Reported Earnings and Retained Quarterly Earnings. Thus, they are difficult for the model to predict and cause large MSEs in the table. Buybacks are also volatile, though not so much so as earnings, as shown in the difficulty of achieving a low MSE in the training and validation data. The MSE of dividends is notably low, but this can be expected due to the consistency with which many companies try to maintain steady, increasing dividends over long periods of time (e.g., the "dividend aristocrats" of Wall Street).

Deeper insights are captured by analyzing the maximum squared error terms for each variable. Table 5.7 summarizes these errors, allowing a comparison of the fits across the data sets, but it also shows how well the model predicts extreme movements in the data. Ninety-five percent prediction intervals were calculated around each of the predicted values for the quarters containing the greatest squared errors, and the last column of the table indicates whether the true data value was contained within this interval. In only one of the six cases did the prediction interval include the actual data value, indicating the model may not fit the real data too well in extreme cases.

Table 5.7: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	3.9730	Jul-01	69.3797	71.3729	[69.70, 73.03]	No
As Reported Earnings	8356.5983	Oct-08	-72.8428	18.5716	[-13.82, 50.96]	No
Dividends	39.2129	Oct-03	49.5908	43.3288	[38.10, 48.55]	No
Buybacks	710.3068	Apr-01	17.8094	44.4609	[16.22, 72.70]	Yes
Retained Quarterly Earnings	66877.3389	Oct-08	-392.5236	-133.9170	[-243.42, -24.40]	No
S&P 500	234.9931	Jul-00	79.0919	63.7624	[51.85, 75.66]	No

Of the five instances where the actual value was outside the predicted interval, some of these points occurred at moments when the market was experiencing volatile swings like booms or crashes, and other errors can be attributed to model fitting errors, as is indicated in the plots for S&P 500 and Buybacks. For example, the first and second quarters of 2001 coincide directly with the beginning of the 2001 recession. This is also when GDP and Buybacks experienced their maximum squared errors respectively. Similarly, As Reported Earnings and Retained Quarterly Earnings experienced their largest squared errors in the fourth quarter of 2008 at the beginning of the Great Recession following the housing crash. The biggest error for the S&P 500 occurred in the summer of 2000, coinciding almost perfectly with the stock market's "dot-com" bubble, during which stock prices climbed to unprecedentedly high levels even though corporate earnings remained relatively low. The largest squared error for dividends in fourth quarter of 2003 on the other hand, appears to have occurred due to seasonal effects rather than economic phenomena. Close examination of the plot for dividends in Figure 5.2 illustrates that higher dividends are paid in the fourth quarter of each year during the escalation of the housing bubble, perhaps reflecting the increase in company profits being paid out to shareholders as annual dividends.

To conclude this portion of the analysis, a discussion regarding the perspective and scope of the variables is appropriate based on the results and insights gained from the preceding coefficients, plots, R-squared values, MSEs, and maximum squared errors.

Since the scope of buybacks, dividends, and earnings are more focused than that of GDP, it is understandable that more volatility would be present in these variables. GDP is perhaps the most macro view of the economy possible, catching all measurable production within the United States, whereas earnings, buybacks, and dividends are specific to only the 500 largest publicly-held companies in the United States. Moreover, As Reported Earnings are subject to many other variables than those included in this model; and dividends, buybacks, and retained quarterly earnings are determined primarily by the individual corporate boards of the constituent firms in the S&P 500. The companies' decisions are determined based on historical and projected business performance, economic conditions, and shareholder expectations. Thus, even though the S&P 500 measures included in the model capture these decisions in the aggregate, they are more directly subject to microeconomic pressures than GDP, which requires extremely large fluctuations in the U.S. economy in order to register an impact.

The disparity between the macro view of GDP and the S&P 500-specific variables is acknowledged here, and it helps to explain some of the large errors present in the results of the model. Yet, the relationships between GDP and the other variables are relevant. As seen in the coefficients of Table 5.2 through Table 5.4, GDP Per Capita has a proportionately large impact on most of the other five variables, but it has particularly strong effect on Retained Quarterly Earnings. This suggests that the decision to retain earnings may be influenced by the performance of the overall economy, perhaps because projected future economic growth necessitates a certain level of reinvestment in research, development, and the expansion of operations. GDP also has a large positive impact on As Reported Earnings, as would be expected since an expanding economy will yield greater earnings to businesses.

Other influential coefficients can be observed in the tables. The largest is that of the S&P 500 on Retained Quarterly Earnings at -0.663. One interpretation of this value is that the level of the S&P 500 has a significant negative influence on companies' decisions to offer buybacks and dividends or reinvest in their company. This notion seems plausible based on the large weighting placed on the Buybacks coefficient in the Retained Quarterly Earnings equation.

Other observations include the value of the coefficients on dividends and GDP. The coefficient for Dividends in the dividends equation seems to confirm the observed practice of dividend-issuing companies continuing to issue dividends. GDP seems most strongly impacted by As Reported Earnings, which seems intuitive; while the S&P 500 has little impact on GDP, which is also a reasonable conclusion given that the stock market is only a reflection of economic performance, not the economy itself.

Also notable in the coefficients is the effect of the federal funds rate on each of the variables. It appears to have the greatest influence on Retained Quarterly Earnings, perhaps suggesting that the decision to retain earnings or seek outside financing is heavily impacted by the costs of borrowing, as determined largely by the interest rates available in the economy. The federal funds rate also has a strong impact on As Reported Earnings, possibly signaling the effect that interest rates have on spending by consumers and other businesses. The coefficients for the Federal Funds Rate on Buybacks and S&P 500 are comparatively large as well, perhaps confirming the suspicion of many stock market observers that low interest rates have led companies to borrow more, not so much to enable business expansion, but to reward shareholders with higher stock prices through stock buybacks.

Finally, it should be noted again that the determined coefficients of the system of differential equations is not a unique solution. As explained in Chapter IV, the coefficients are only "particular constants" for this problem, and they cannot be applied generally to other problems. Nevertheless, the relationships implied by their values are important, and they serve to illustrate the effect that one variable has on another.

5.3.2 Unweighted Model

The model's objective function, equation (4.14), includes a weighting, $\frac{t}{T}$, that causes the model to consider later inputs more heavily than early inputs. That is, as the model progresses from the first quarter of 1998 to the third quarter of 2011, economic inputs to the model are given more weight as t increases toward T. If this weighting is removed, each residual in the model receives equal weighting and minimizing the sums of squared errors produces different results. This section discusses the results of fitting the unweighted model to the data points.

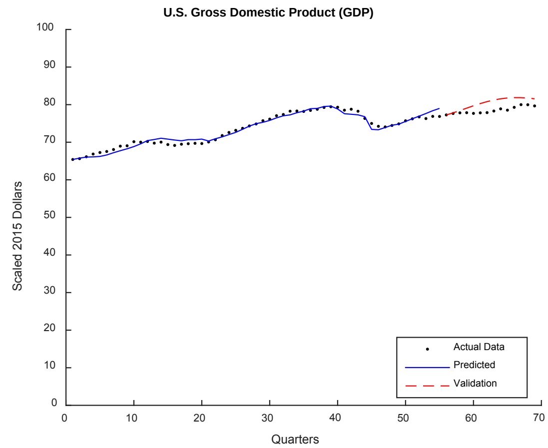
As seen in Table 5.8, fitting the unweighted model produces distinct results compared to those obtained in Section 5.3.1. In the variables that are relatively stable, such as GDP Per Capita and Dividends, the MSEs tend to remain the same or decrease slightly compared to the unweighted MSEs obtained from the weighted model. GDP Per Capita increases by only 0.007 percentage points in the training set, but it decreases by more than 11 points in the validation set. Likewise, whereas the MSE for Dividends increases by 0.99 in the training set, it decreases by a factor of two in the validation set.

Table 5.8: Mean Square Errors from Unweighted Model

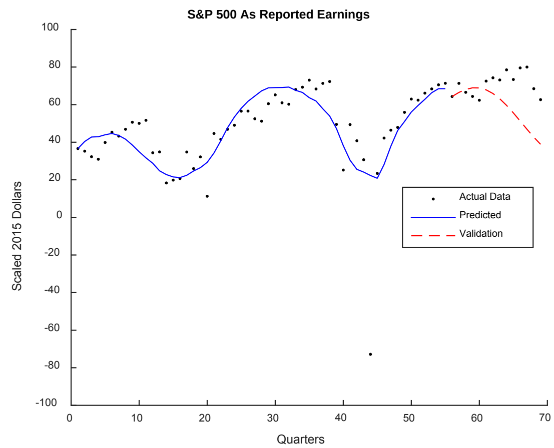
	Training Set Mean Square Errors	Validation Set Mean Square Errors	Difference from Weighted Model	
GDP Per Capita	0.67635	4.58052	0.00711	-11.69906
As Reported Earnings	244.91567	291.08277	-7.17246	256.05050
Dividends	7.64757	14.45242	0.98770	-28.55202
Buybacks	34.03497	171.64596	-158.31924	14.13082
Retained Quarterly Earnings	2730.70311	926.58338	-149.70351	-1124.75418
S&P 500	56.28577	101.46491	22.32280	-95.99457
Total	3074.26344	1509.80996	-291.87760	-990.81849

More dramatic differences are apparent in the other variables. The error in the As Reported Earnings validation set increases significantly by 256 points, whereas the MSE for the validation set of Retained Quarterly Earnings decreases by 1,124. MSE for the S&P 500 increases only in the training set while decreasing by nearly 50% in the validation set. This decrease may not be meaningful however, since the model seems to trend in the wrong direction after initially improving, as shown in Figure 5.3.

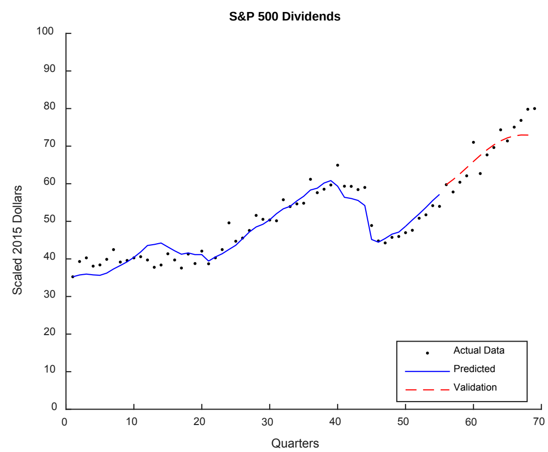
Elsewhere, the plots seem to indicate improved performance in trends compared to the weighted model. The fit in the Buybacks, GDP, and Dividends curves looks markedly improved, although the trend of the validation data again seems to drop too quickly compared to the actual data.



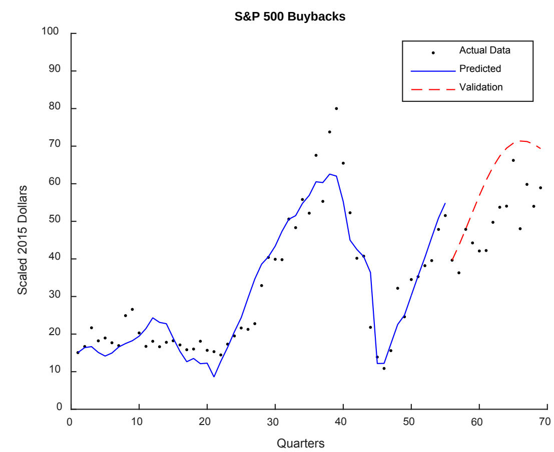
(a) GDP Per Capita



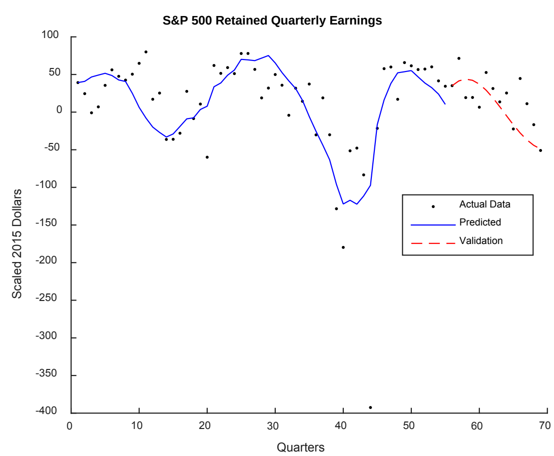
(b) As Reported Earnings



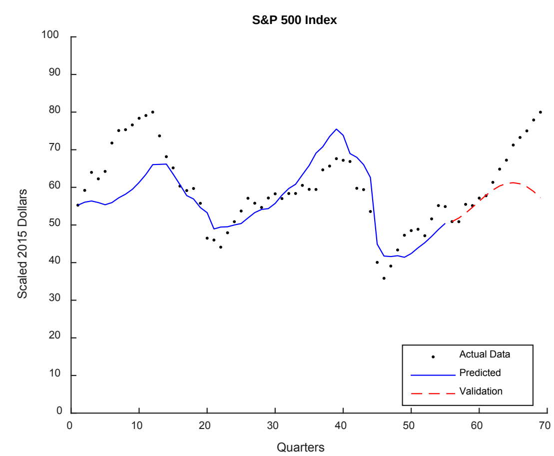
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.3: Plots showing fitted training data and validation data

Table 5.9 summarizes the maximum squared errors and the prediction intervals on the predicted data points for those quarters. The results are fairly similar to those in the weighted model, with the most extreme errors appearing in the fourth quarter of 2008 on As Reported Earnings and Retained Quarterly Earnings. As was the case with most components of the weighted model, the actual values do not fall within the 95% prediction intervals for the predicted data points, although some are very close the boundaries of the intervals.

Table 5.9: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	4.6285	Jul-11	76.8530	79.0044	[77.29, 80.70]	No
As Reported Earnings	9052.1022	Oct-08	-72.8428	22.2997	[-9.63, 54.23]	No
Dividends	49.9547	Oct-03	49.5908	42.5230	[36.92, 48.11]	No
Buybacks	322.7391	Jul-07	80.0000	62.0351	[50.18, 73.88]	No
Retained Quarterly Earnings	87178.1272	Oct-08	-392.5236	-97.2641	[-203.88, 9.36]	No
S&P 500	319.4855	Jul-99	75.0920	57.2178	[41.81, 72.62]	No

The R-squared values for the fitted curves are provided in Table 5.10. Interestingly, here the computed R-squared values indicate that Dividends and Buybacks are the best fit, as was noted already in the case of Buybacks, and all of the modeled variables again have values over 0.5. GDP Per Capita registers an R-squared value greater than one because the system of differential equations injected more variability into the predicted data than was present in the actual data, which is quite smooth relative to its mean. Since R-squared is normally used in linear regression and is the ratio of variation in the model compared to variation in the actual data, the measure inherently assumes that a model will only match the variation present in the data but never exceed it. In regression, this assumption is valid because the model is first based on the mean of the data, then fit further until the SSE is minimized. Differential equations and nonlinear optimization are not fitting a straight line to a set of data points however. They use numerical methods to incrementally fit predicted points to the true points as closely as possible, occasionally resulting in increased variability in the modeled curves, as is the case here. Thus, in this model, the R-squared value should be considered in conjunction with the other measures of fit and the plots themselves. If the curves closely parallel the actual data and yield R-squared values close to one, the R-squared values reflect a good model fit.

Table 5.10: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	1.0160
As Reported Earnings	0.5238
Dividends	0.9351
Buybacks	0.9171
Retained Quarterly Earnings	0.5071
S&P 500	0.7649

The coefficients for this solution are provided in Table 5.11 through Table 5.13, and the same observations made in Section 5.3.1 regarding the model coefficients can be made here, where positive values reflect a complementary effect of variable i on variable j in the differential equations. Likewise, negative values denote a substitutionary effect.

Table 5.11: *A*-coefficients in matrix form

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.00483	0.00350	0.00078	-0.00184	0.00348	0.00091
As Reported Earnings	-0.02683	-0.01143	0.01396	-0.07938	0.00870	-0.19648
Dividends	0.00927	0.01040	0.00023	0.00171	0.00797	0.00234
Buybacks	0.03360	0.01960	0.00816	-0.00229	0.02357	-0.01089
Retained Quarterly Earnings	-0.14310	0.02688	0.06392	-0.23321	-0.06302	-0.13356
S&P 500	-0.00528	0.02157	-0.00100	0.00995	0.01259	-0.00982

Table 5.12: *B*-coefficients in matrix form

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.49948	0.49608	0.49855	0.50106	0.48984	0.49460
As Reported Earnings	0.53251	0.55680	0.48309	0.84168	0.50223	0.68805
Dividends	0.49662	0.46023	0.50246	0.48617	0.47269	0.49349
Buybacks	0.29283	0.63538	0.56478	0.50655	0.42561	0.14553
Retained Quarterly Earnings	0.22743	1.00000	0.04297	0.20773	0.45080	0.27485
S&P 500	0.26698	0.38945	0.51845	0.39025	0.50235	0.76863

Table 5.13: *D*-coefficients in matrix form

	Federal Funds Rate
GDP Per Capita	0.00133
As Reported Earnings	0.00000
Dividends	-0.00018
Buybacks	-0.00175
Retained Quarterly Earnings	0.00324
S&P 500	0.06862

Although weighting the model is not beneficial across all variables, it does improve the training data performance of several model components, including the S&P 500, which is the primary variable of interest. Particularly, the R-squared values improve for several of the model components. Since the later residuals are not weighted in this version of the model, the predicted curves are fit to the data over the entire range of the training set, resulting in better fits overall. This also results in lower MSEs for some of the variables, even if maximum squared errors tend to increase for most of the curves. Yet, the trend of the validation curves appears slightly better in the weighted model. In the weighted model, the weights, $\frac{t}{T}$, were added in order to more accurately predict future system behavior in the validation period. Since the weighted model seems to accomplish this slightly better than the unweighted model, further analysis using the data points continued with the weights in place.

5.4 Fitting the Model to the Data Slopes

Fitting the model based on minimizing the differences in slopes pursues a separate objective than fitting to the data points themselves. Although the residuals on the fitted slopes may be larger than those calculated from the fitted data, the trend in the slopes may be more accurate. As discussed in Chapter IV, the residuals between the actual slopes in the data and the fitted curves were calculated by taking the difference between the actual slopes and the calculated values of the differential equations for each variable at each time step. The sum of squared errors was then found by squaring these differences and summing the values across the entire data set. This sum of squares was then divided by the number of observations to get the mean squared error. Minimizing the SSE for the slopes yielded the coefficients in Table 5.14 through Table 5.16.

Table 5.14 *A*-coefficients in matrix form calculated based on slopes

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.00721	0.01874	0.00160	-0.01683	-0.00207	-0.00544
As Reported Earnings	0.02912	-0.01584	-0.05342	-0.20823	-0.01692	-0.56616
Dividends	0.01530	0.04128	-0.08163	0.00272	-0.00152	0.00374
Buybacks	0.05965	0.01516	-0.02058	-0.02026	0.02347	-0.05291
Retained Quarterly Earnings	-5.35666	-0.05821	0.01402	-0.05658	-0.68419	-3.24459
S&P 500	-0.02046	0.08454	-0.03351	-0.04240	-0.01059	-0.01234

Table 5.15 *B*-coefficients in matrix form calculated based on slopes

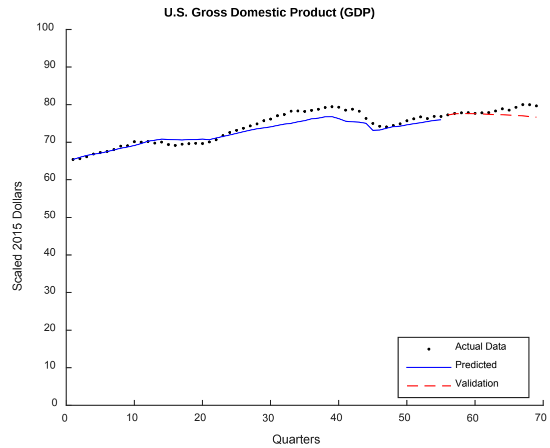
	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.51728	0.55442	0.49735	0.54754	0.46065	0.52311
As Reported Earnings	0.82311	0.93421	0.11067	1.00000	0.11123	0.88526
Dividends	0.48315	0.48292	0.49019	0.48215	0.48102	0.46451
Buybacks	0.49058	0.69096	0.49865	0.32674	0.57351	0.60195
Retained Quarterly Earnings	0.69179	1.00000	0.05487	1.00000	0.99873	0.99991
S&P 500	0.41889	0.45806	0.46543	0.56210	0.47715	0.35620

Table 5.16: *D*-coefficients in matrix form calculated based on slopes

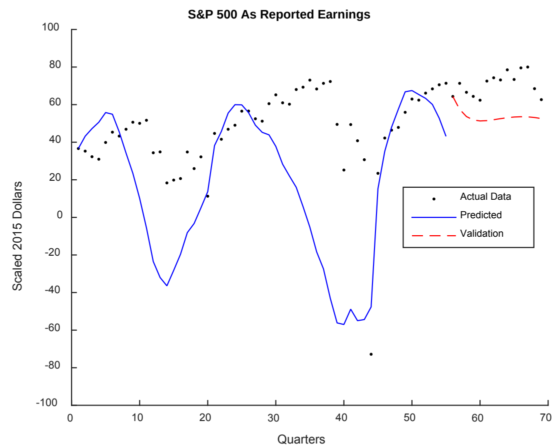
	Federal Funds Rate
GDP Per Capita	0.01799
As Reported Earnings	-0.00697
Dividends	-0.01310
Buybacks	-0.00033
Retained Quarterly Earnings	0.59911
S&P 500	0.01675

Applying these coefficients across the training and validation data sets produced a worse fit than was obtained by fitting to the data alone, as can be seen in the plots of the fitted curves in Figure 5.4 below. In contrast to the previous model fits in Sections 5.3.1 and 5.3.2, the predicted curves here fail to match the data values. The training data curves do seem to match the trends of the actual data, but these trends are greatly exaggerated. The trends in the validation sets are not very accurate, with several of them, including the S&P 500, angling downward when the actual data slopes sharply upward.

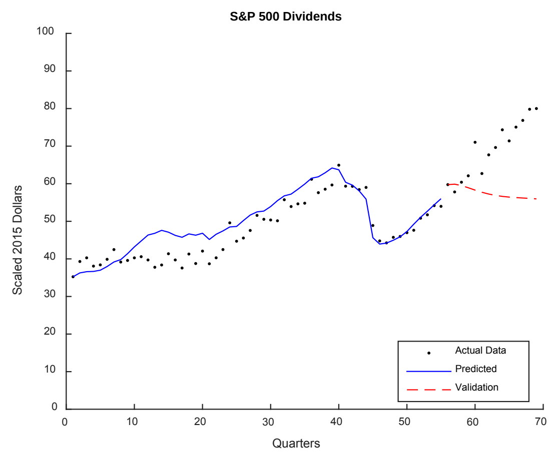
The poor data fit is also apparent in the MSEs. Each of the training sets increased by substantial margins compared to the MSEs computed by minimizing the SSEs on the data, even if some improvements are made in the validation data sets. Table 5.17 gives these results, and the differences to the MSEs in Table 5.6. Positive differences in the table indicate an increase in MSE and negative differences indicate a decrease. Similar to the findings in Section 5.3.1, the values listed in the table are the unweighted residuals even though the weighted model, equation (4.14) with the term $\frac{t}{T}$, was used to minimize the sum of squared residuals.



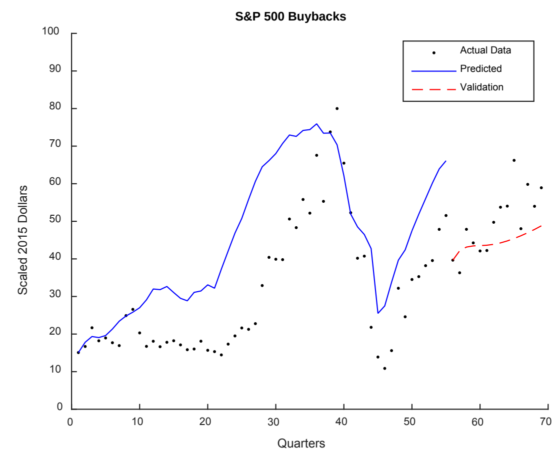
(a) GDP Per Capita



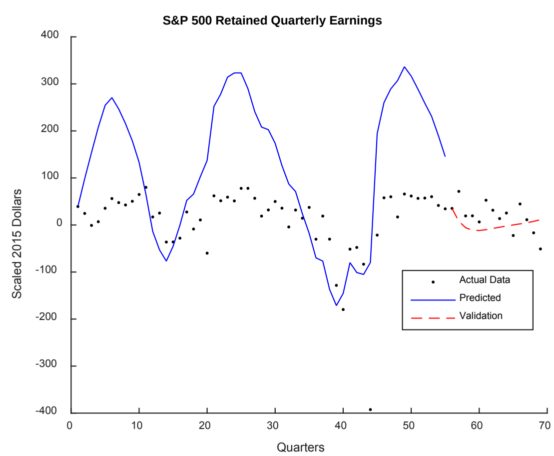
(b) As Reported Earnings



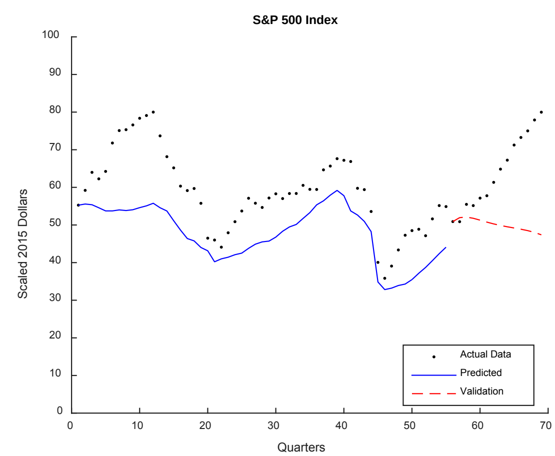
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.4: Plotted training and validation curves

Table 5.17 Weighted Mean Square Errors Based on Slopes

	Training Set Mean Square Errors	Validation Set Mean Square Errors	Difference from Model Data Fit	
GDP Per Capita	2.5306	2.7176	1.86139	-13.56200
As Reported Earnings	2152.7059	340.3568	1900.61775	305.32449
Dividends	16.0997	209.0925	9.43982	166.08806
Buybacks	307.1737	72.4354	114.81948	-85.07976
Retained Quarterly Earnings	26065.1028	1374.9384	23184.69619	-676.39913
S&P 500	147.8436	321.7648	113.88067	124.30531
Total	28691.4563	2321.3054	25325.31530	-179.32302

As shown in Table 5.17, minimizing the least squares model with respect to the slopes in the training data results in a worse fit than if the model is fit to the data points themselves. The major exceptions to this conclusion can be seen in the validation set for Retained Quarterly Earnings and Buybacks. In each of these variables, the validation set improved substantially, which is likely due to the tempered behavior of the curves within the validation region.

The maximum squared errors in Table 5.18 shows much wider intervals than those produced from the data models, mainly due to the larger SSEs of these data sets. In this case, the actual values for Retained Quarterly Earnings and S&P 500 lie inside the prediction intervals for the worst predicted point, indicating the model may predict large movements in these components better than the data model. As mentioned however, this improvement can mostly be attributed to the large swings in the predicted data itself and likely does not evidence greater prediction accuracy.

Table 5.18: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	11.1596	Apr-08	78.7812	75.4406	[72.20, 78.67]	No
As Reported Earnings	13301.1369	Apr-07	72.3027	-43.0279	[-137.18, 51.12]	No
Dividends	85.0350	Apr-01	38.3781	47.5996	[39.42, 55.76]	No
Buybacks	1433.6013	Jul-04	22.7718	60.6347	[25.19, 96.07]	No
Retained Quarterly Earnings	97890.8517	Oct-08	-392.5236	-79.6484	[-409.07, 249.77]	Yes
S&P 500	586.6836	Oct-00	80.0000	55.7784	[30.96, 80.58]	Yes

Lastly, the same findings are obtained from the R-squared values. All of the variables fall below 0.5, except for Retained Quarterly Earnings at 0.5177, indicating the model fit does not improve if the difference between the slopes is minimized.

Table 5.19: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	0.4528
As Reported Earnings	0.3430
Dividends	0.3692
Buybacks	0.3135
Retained Quarterly Earnings	0.5177
S&P 500	0.3667

5.4.1 Unweighted Model Based on Slopes

Similar to the unweighted model for the data, minimizing the unweighted sums of squared residuals based on the slopes produced distinct results compared to the weighted model for slopes. First, the coefficients for these results are displayed in Table 5.20 through Table 5.22.

Table 5.20 *A*-coefficients in matrix form calculated based on slopes

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.00468	0.01149	-0.01952	-0.00409	-0.00136	-0.00668
As Reported Earnings	0.02119	-0.02752	-0.04883	-0.15338	-0.06272	-0.22729
Dividends	0.11884	0.02299	-0.19290	0.06832	-0.00053	-0.02945
Buybacks	0.02980	-0.03048	0.15293	-0.03711	0.03654	-0.03316
Retained Quarterly Earnings	-1.93792	-0.02722	0.34679	-0.90408	-0.59365	-1.55834
S&P 500	0.00098	0.08572	-0.06096	-0.03833	-0.01143	-0.10786

Table 5.21 *B*-coefficients in matrix form calculated based on slopes

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.50209	0.49769	0.50247	0.50016	0.50058	0.50136
As Reported Earnings	0.61522	0.51097	0.49173	0.53655	0.52059	0.53955
Dividends	0.42498	0.51390	0.38402	0.55217	0.50022	0.52414
Buybacks	0.45208	0.49574	0.53920	0.49281	0.51198	0.53247
Retained Quarterly Earnings	0.44879	1.00000	0.24361	1.00000	1.00000	1.00000
S&P 500	0.48735	0.49986	0.55083	0.51125	0.55215	0.46783

Table 5.22: *D*-coefficients in matrix form calculated based on slopes

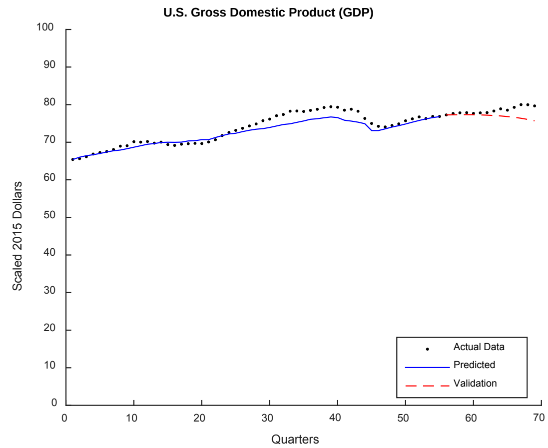
	Federal Funds Rate
GDP Per Capita	0.01019
As Reported Earnings	0.00000
Dividends	-0.00063
Buybacks	0.00000
Retained Quarterly Earnings	0.11994
S&P 500	0.21017

Solving the unweighted model for slopes shows that while its performance improved over the weighted model for slopes, it does not perform as well against the baseline case of the weighted model for data. This is especially true for the more volatile variables, As Reported Earnings and Retained Quarterly Earnings, which increase by 879 and 7,960 points respectively, detailed in Table 5.23.

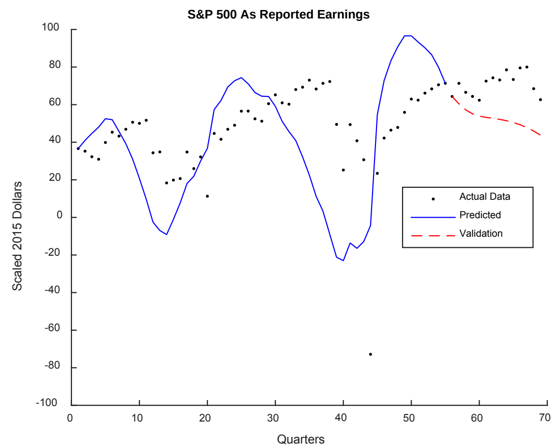
Table 5.23: Results from Unweighted Model Based on Slopes

	Training Set Mean Square Errors	Validation Set Mean Square Errors	Difference from Weighted Model Based on Slopes		Difference from Weighted Model Based on Data	
GDP Per Capita	2.4632	4.3403	-0.0674	1.6227	1.7940	-11.9393
As Reported Earnings	1131.9915	407.9906	-1020.714	67.6339	879.9033	372.9584
Dividends	9.7846	169.0470	-6.3151	-40.0455	3.1247	126.0425
Buybacks	69.7311	60.7330	-237.4426	-11.7024	-122.6231	-96.7821
Retained Quarterly Earnings	10840.5396	1075.2016	-15224.56	-299.7368	7960.1330	-976.1360
S&P 500	60.1285	576.5766	-87.7151	254.8118	26.1655	379.1172
Total	12114.6385	2293.8891	-16576.82	-27.4163	8748.4974	-206.7393

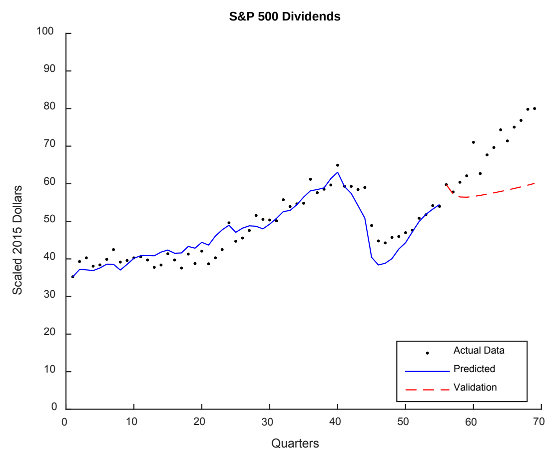
The improvement over the weighted model for slopes is readily apparent in the series of plots shown in Figure 5.5. In these, the objective of matching the trends in the true data is definitely accomplished in Buybacks and Retained Quarterly Earnings, and to a lesser extent in GDP, As Reported Earnings, and Dividends. The trend of the validation sets across all these variables seem to be lacking however, often going in a direction opposite of the actual data.



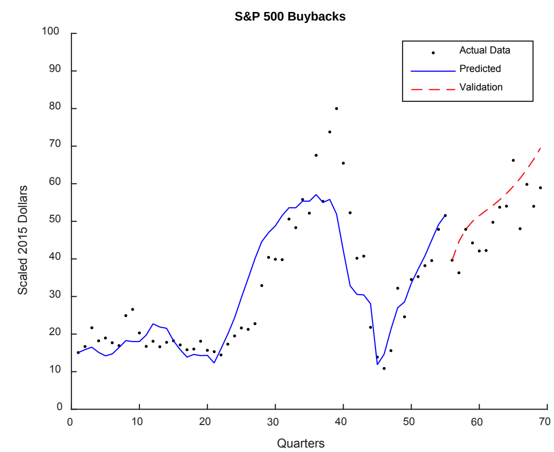
(a) GDP Per Capita



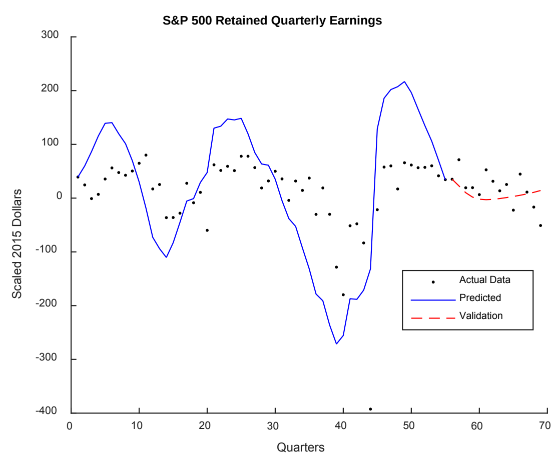
(b) As Reported Earnings



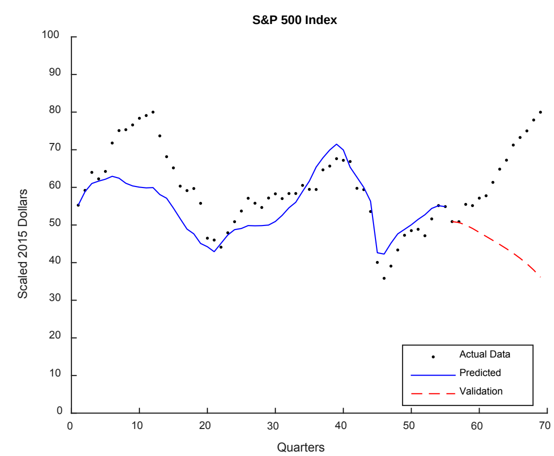
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.5: Plotted training and validation curves after solving the system of differential equations using the weighted data slopes

Table 5.24 summarizes the maximum squared errors resulting from this approach. Like the weighted version of model for slopes, the prediction intervals here do not overlap with the actual data, illustrating the poor predictive value.

Table 5.24: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	11.2775	Jan-06	78.2603	74.9021	[71.72, 78.07]	No
As Reported Earnings	6603.4118	Apr-07	72.3027	-8.9587	[-77.23, 59.31]	No
Dividends	71.6324	Jan-09	48.8840	40.4204	[34.03, 46.81]	No
Buybacks	787.2623	Jul-07	80.0000	51.9418	[34.98, 68.90]	No
Retained Quarterly Earnings	68095.0311	Oct-08	-392.5236	-131.5733	[-344.01, 80.87]	No
S&P 500	402.2573	Oct-00	80.0000	59.9436	[44.12, 75.76]	No

Likewise, the R-squared values in Table 5.25 are low. None of the values reach 0.5, indicating the model is not fit well to the data by minimizing the SSEs of the unweighted slopes.

Table 5.25: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	0.4254
As Reported Earnings	0.2638
Dividends	0.4900
Buybacks	0.3061
Retained Quarterly Earnings	0.3765
S&P 500	0.4900

5.5 Fitting the Model to the Data Secants

By fitting the model to the data secants, it is possible to capture trends more accurately than is available from the data or slopes alone. The secants for the data were first calculated according to the procedure outlined in Section 4.3.3, using equation (4.6). The secants for the fitted curves were calculated in a similar manner. The predicted data points were found using the normal method, where the differential equations are calculated by taking inputs from the actual data (for the training set) or from the previously computed predicted values (for the validation set). The secants for the predicted values were then calculated using equation (4.6), and the differences between the predicted secants and the actual secants were minimized according to the model in (4.14) – (4.19).

Minimizing the weighted model for the secants produces the following set of coefficients, given in Table 5.26 through Table 5.28.

Table 5.26 *A*-coefficients in matrix form calculated based on weighted secants

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.00724	0.02031	0.00159	-0.01688	-0.00199	-0.00543
As Reported Earnings	0.03343	0.00068	-0.06142	-0.08538	-0.01517	-0.56770
Dividends	0.01490	0.04095	-0.06813	0.00277	-0.00158	0.00373
Buybacks	0.04955	0.03916	-0.02564	-0.01620	0.02125	-0.06034
Retained Quarterly Earnings	-0.49008	-0.09276	-0.13530	-0.31040	-0.41189	-2.80796
S&P 500	-0.02104	0.08617	-0.03435	-0.04258	-0.01002	-0.01282

Table 5.27 *B*-coefficients in matrix form calculated based on weighted secants

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.51632	0.54270	0.49959	0.53987	0.48154	0.52296
As Reported Earnings	0.65930	0.95410	0.13047	1.00000	0.21834	1.00000
Dividends	0.49651	0.50088	0.49515	0.47404	0.45922	0.52914
Buybacks	0.45494	0.53547	0.42442	0.51104	0.55593	0.54076
Retained Quarterly Earnings	0.27590	1.00000	0.14678	1.00000	1.00000	1.00000
S&P 500	0.44301	0.44495	0.45076	0.53798	0.52189	0.37521

Table 5.28: *D*-coefficients in matrix form calculated based on weighted secants

	Federal Funds Rate
GDP Per Capita	0.01804
As Reported Earnings	-0.00612
Dividends	-0.01318
Buybacks	-0.00033
Retained Quarterly Earnings	1.16564
S&P 500	0.01623

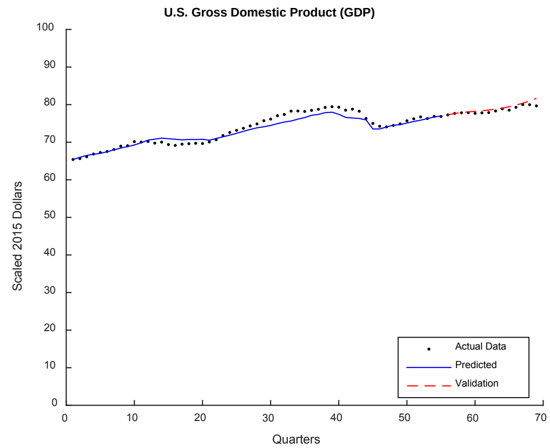
When these coefficients are applied across the training and validation data sets, the plots in Figure 5.6 are generated. Like the curves resulting from the data, the overall fit of the training data parallels the behavior of the actual data. Yet, some of the validation trends are incorrect, such as those on Buybacks and Dividends where the curves slope downward rather than following the actual data upward. Thus, this model's predictive value is limited.

Comparing these curves to the fits obtained for the slopes in the previous section reveals that the model fits better to the secants than to the slopes. This observation is more obvious considering the quantitative results shown in Table 5.29. The unweighted MSEs are given in the second and third columns, and the differences between these scores and the baseline weighted data model are shown in the two right-most columns.

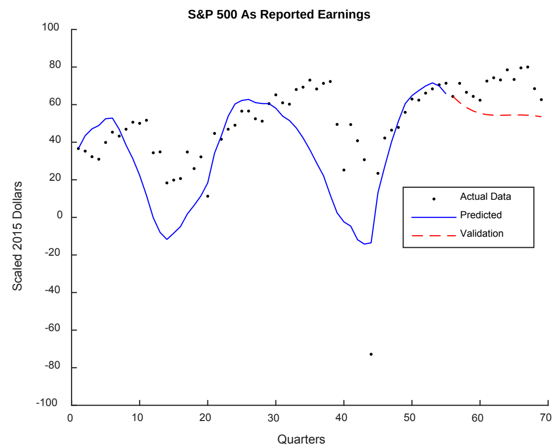
Table 5.29: Weighted Mean Square Error Based on Weighted Secant Model

	Training Set Mean Square Errors	Validation Set Mean Square Errors	Difference from Model Data Fit	
GDP Per Capita	1.35176	0.56038	0.68252	-15.71920
As Reported Earnings	663.54608	275.90585	411.45794	240.87358
Dividends	6.96792	131.76118	0.30805	88.75674
Buybacks	43.40761	281.62013	-148.94660	124.10500
Retained Quarterly Earnings	5536.43077	6133.92172	2656.02415	4082.58416
S&P 500	71.54651	75.44696	37.58355	-122.01251
Total	6323.25065	6899.21622	2957.10961	4398.58777

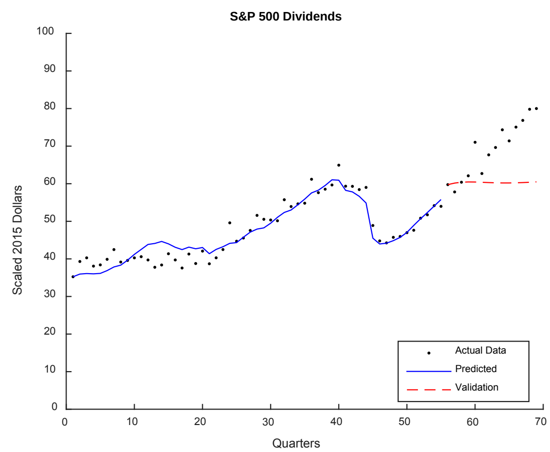
Like fitting the model based on the slopes, the quantitative MSEs from the secant model show a poorer fit than that obtained from the data points themselves, with large increases in nearly every variable in both the training and validation sets. Although improvement is observed in the validation data for Retained Quarterly Earnings, the errors in this variable still remain large, and all other variables show a decrease in performance in both the training and validation sets. These observations indicate the weighted data model is superior to the secant model.



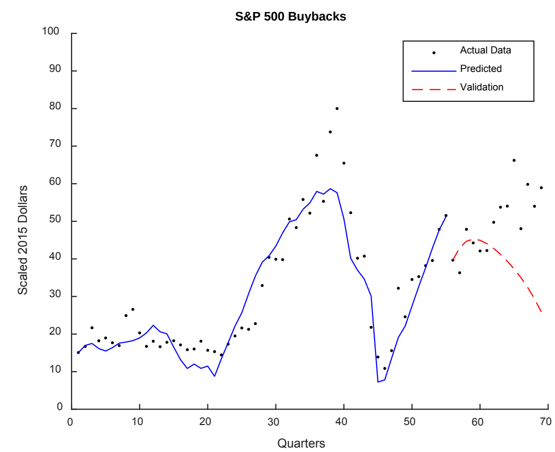
(a) GDP Per Capita



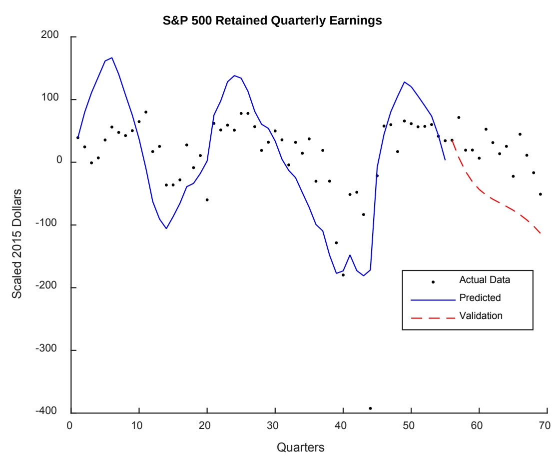
(b) As Reported Earnings



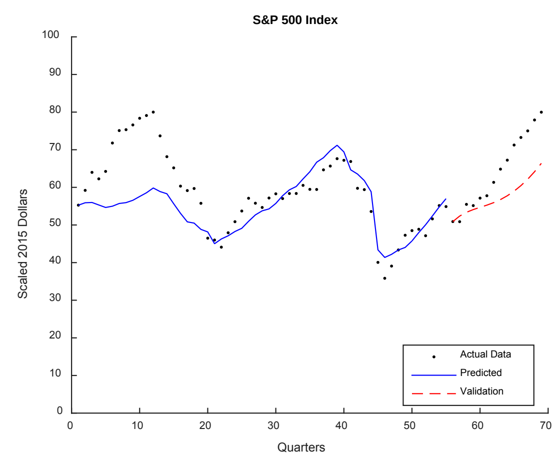
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.6: Plotted training and validation curves

The maximum squared errors of Table 5.30 illustrate this further. While most of the errors are smaller than those obtained in the model for slopes, the maximum squared errors are higher than those in the weighted data model, except in the cases of As Reported Earnings and Retained Quarterly Earnings. Moreover, none of the actual values fall within the 95% prediction interval for the predicted data value, indicating the model does not predict extreme events very well, and the model will likely not accurately forecast a future extreme event. This shortcoming is common to all three model-fitting approaches.

Table 5.30: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	6.8178	Jan-06	78.2603	75.6492	[73.29, 78.00]	No
As Reported Earnings	3647.6376	Apr-07	72.3027	11.9070	[-40.36, 64.18]	No
Dividends	40.1494	Jan-01	37.7717	44.1080	[38.72, 49.48]	No
Buybacks	501.0724	Jul-07	80.0000	57.6154	[44.23, 70.99]	No
Retained Quarterly Earnings	48810.0501	Oct-08	-392.5236	-171.5936	[-323.41, -19.77]	No
S&P 500	432.3419	Apr-00	78.3628	57.5699	[40.27, 74.86]	No

Finally, the R-squared values of Table 5.31 indicate that while the model for weighted secants performs better than both the weighted and unweighted models for slopes, the fit is generally not as good as that obtained by the weighted model for data.

Table 5.31: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	0.6509
As Reported Earnings	0.3230
Dividends	0.5567
Buybacks	0.6180
Retained Quarterly Earnings	0.4574
S&P 500	0.5041

5.5.1 Unweighted Model Based on Secants

Assessing the performance of the unweighted secant model indicates it performs better than the weighted model, although the weighted model for data is still superior. The coefficients for these results are displayed in Table 5.32 through Table 5.34.

Table 5.32 *A*-coefficients in matrix form calculated based on unweighted secants

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.00474	0.00435	-0.00469	-0.00169	0.00166	-0.00115
As Reported Earnings	-0.03468	-0.02677	-0.06101	-0.05557	-0.02066	-0.40775
Dividends	0.02071	0.01623	-0.05267	0.01313	0.00358	-0.00114
Buybacks	0.03886	0.02350	0.00025	-0.01409	0.02511	-0.05809
Retained Quarterly Earnings	-0.25042	-0.25670	-0.09213	-0.33731	-0.13522	-1.09988
S&P 500	0.00854	0.02533	-0.01007	0.00043	0.00905	-0.04429

Table 5.33 *B*-coefficients in matrix form calculated based on unweighted secants

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.51676	0.48179	0.50887	0.50374	0.47227	0.50383
As Reported Earnings	0.62180	0.64373	0.20558	1.00000	0.47688	0.92424
Dividends	0.30924	0.41025	0.32635	0.28012	0.57247	0.49516
Buybacks	0.47442	0.46486	0.50021	0.53736	0.56685	0.56580
Retained Quarterly Earnings	0.36706	0.66757	0.41607	0.90031	0.54287	0.72369
S&P 500	0.52025	0.37177	0.49760	0.47925	0.50150	0.47866

Table 5.34: *D*-coefficients in matrix form calculated based on unweighted secants

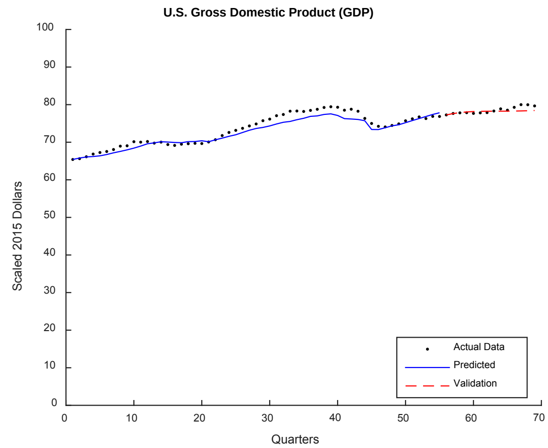
	Federal Funds Rate
GDP Per Capita	0.00328
As Reported Earnings	0.00000
Dividends	-0.00695
Buybacks	-0.00009
Retained Quarterly Earnings	0.49892
S&P 500	0.07834

The plots of Figure 5.7 are similar to those of Figure 5.6 in that the unweighted secants tend to match the trends of the training data quite well; the fit is very good for GDP, Dividends, Buybacks, and the S&P 500. Problems are again evident however in the validation sets where the curves tend to deviate drastically from the direction of the actual data.

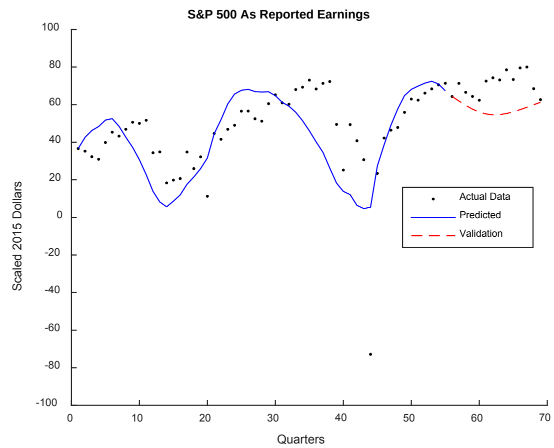
Again, like with the model for the unweighted slopes, improvement is noted in the MSEs for the unweighted secants over the weighted secant model, showing that the unweighted secant model likely outperforms the weighted model. Yet, overall both secant models perform poorly in nearly every component compared to the weighted model for data. The MSE results are summarized in Table 5.35.

Table 5.35: Unweighted Mean Square Error Based on Unweighted Secant Model

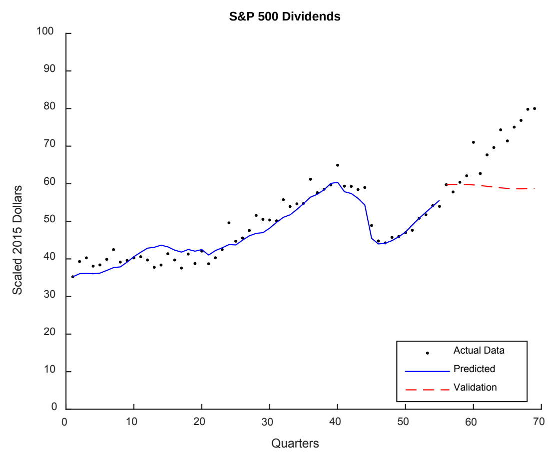
	Training Set Mean Square Errors	Validation Set Mean Square Errors	Difference from Weighted Model		Difference from Weighted Model Based on Data	
GDP Per Capita	1.65602	0.63377	0.30426	0.07340	0.98678	-15.64580
As Reported Earnings	400.65181	222.82246	-262.89426	-53.08339	148.56368	187.79019
Dividends	7.05866	158.83506	0.09074	27.07388	0.39880	115.83062
Buybacks	44.03278	56.19644	0.62517	-225.42369	-148.32143	-101.31869
Retained Quarterly Earnings	3925.60181	2267.72042	-1610.82896	-3866.20130	1045.19519	216.38286
S&P 500	78.27636	286.83135	6.72985	211.38439	44.31340	89.37188
Total	4457.27745	2993.03951	-1865.97320	-3906.17671	1091.13641	492.41106



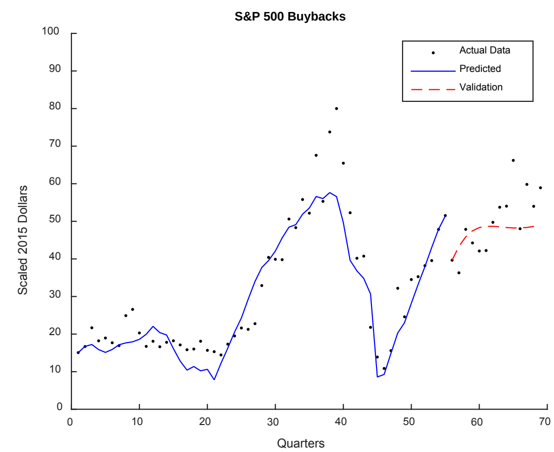
(a) GDP Per Capita



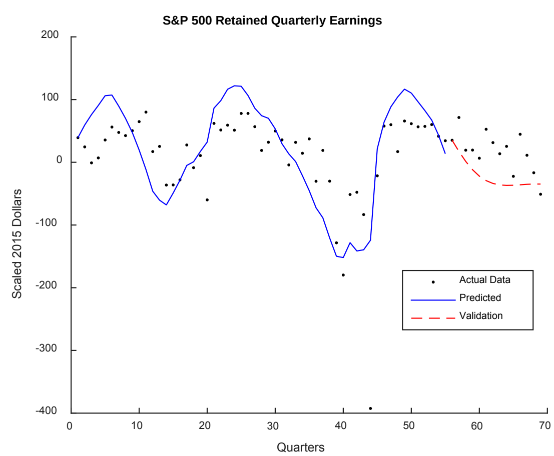
(b) As Reported Earnings



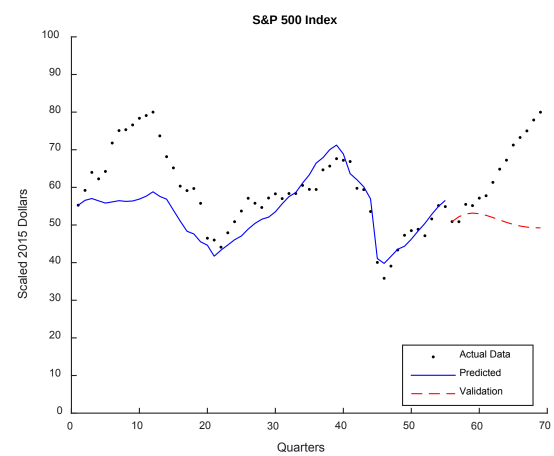
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.7: Plotted training and validation curves

Table 5.36 gives the maximum squared errors for the unweighted secant training data. The largest errors are again associated with As Reported Earnings and Retained Quarterly Earnings, and none of the prediction intervals contain the true data values. Some of the prediction intervals, such as GDP, Dividends, and S&P 500, come very close to overlapping with the actual data value, showing that this approach has real merit, and improvements on the model may add to its predictive ability.

Table 5.36: Maximum squared errors of training data, with prediction intervals

	Maximum Squared Error	Quarter	Actual Data Value	Predicted Data Value	Prediction Interval	Overlap?
GDP Per Capita	7.4500	Jan-06	78.2603	75.5308	[72.92, 78.13]	No
As Reported Earnings	6119.7271	Oct-08	-72.8428	5.3859	[-35.45, 46.22]	No
Dividends	33.3231	Oct-03	49.5908	43.8182	[38.44, 49.19]	No
Buybacks	549.0675	Jul-07	80.0000	56.5678	[43.09, 70.04]	No
Retained Quarterly Earnings	71932.2975	Oct-08	-392.5236	-124.3216	[-252.16, 3.52]	No
S&P 500	461.2091	Apr-00	78.3628	56.8870	[38.79, 74.98]	No

The R-squared results in Table 5.37 are slightly improved from the weighted secant model, although, at a maximum value of only 0.6127, these fits are poor compared to the established baseline of the weighted data model.

Table 5.37: Training data R-squared values

	Training Set R-Squared
GDP Per Capita	0.6127
As Reported Earnings	0.2087
Dividends	0.5114
Buybacks	0.5808
Retained Quarterly Earnings	0.2935
S&P 500	0.5591

Directly comparing the results of the model when it is solved using the data points, slopes, and secants shows that the model performs best when fitted to the data itself. Furthermore, the weighted model seems to return smaller errors than the unweighted model in the training data. The unweighted model did perform better in the validation set, but these improvements were marginal for most of the variables. The most volatile variables seemed to benefit from the slope and secant models, especially in the unweighted cases, but these improvements were not significant enough to conclude that these models perform better than the baseline. Recognizing this, further analysis will focus on the weighted data model.

5.6 Residual Analysis

Ideally, the unweighted residuals of the predicted values are normally distributed, meaning that the model errs in a random fashion, predicting values both higher and lower than the actual data with equal probability. This was verified on the weighted data model using normal probability plots. Large deviations from the center diagonal line indicate departures from normality, but do not necessarily invalidate the model or its results. As Vining explains in regards to regression analysis, the residuals do not need to be strictly normally distributed they only need to conform adequately to well-behaved distributions. Most statistical inference tests are robust to mild departures from normality (Vining, 2011:107).

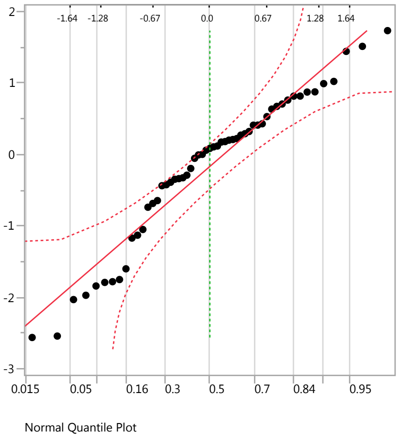
This is also true for dynamical models, where normally-distributed residuals indicate adherence to mean field theory. According to mean field theory, many independent variables can be approximated by the mean behavior of all the variables together (Opper & Saad, 2001:ix). Thus, residuals that are reasonably normally-distributed indicate model adequacy since system behavior is largely approximated by the mean values produced by the model.

The residual analysis used the externally studentized residuals calculated via the procedures and formulas outlined in Section 4.6.1. As shown in Figure 5.8, the residuals from each economic variable appear normal over the training data. Where deviations do occur, the plotted residuals mostly remain inside of the 95% confidence intervals. The largest deviations are observed on As Reported Earnings and Retained Quarterly Earnings, but the volatility of these indexes has already been discussed, and the largest deviations in these plots occur due to single outliers representing the economic recession and stock market crash of the 4th quarter of 2008.

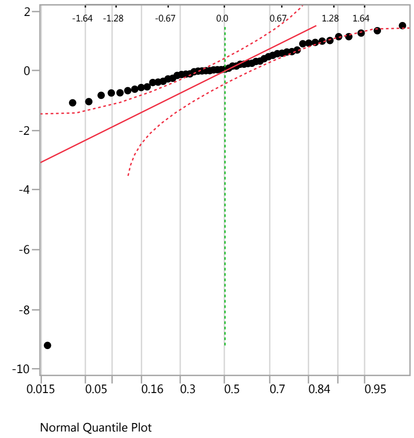
The goodness-of-fit corresponding to the normal probability plot for each economic variable was calculated as well. As seen in Table 5.38, the residuals for Dividends are easily identified as normally distributed, while the others have *p*-values below 0.05, indicating they are not normal. However, as already mentioned for As Reported Earnings and Retained Quarterly Earnings, the low *p*-values associated with these variables are due to the single outlier from 2008. The remaining residuals on these plots are normally distributed with no drastic deviations. GDP and S&P 500 residuals show slight deviations, with no major outliers, but nearly all the data points are within the 95% confidence interval. The low *p*-values in these cases are likely due to the meandering pattern of the residuals in Figure 5.8 that grows larger towards the tails. Neither these departures nor the others are cause for alarm however. The Shapiro-Wilk test is conservative in its assessment of normality, and the normal probability plots provide evidence to conclude the data is generally well-behaved.

Table 5.38: Goodness-of-Fit test results for residual analysis of the weighted least squares model

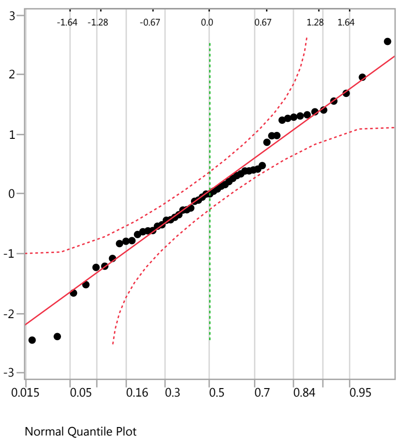
Shapiro-Wilks Test for Goodness-of-Fit	*P*-Value
GDP Per Capita	0.0193
As Reported Earnings	< 0.0001
Dividends	0.6734
Buybacks	0.0561
Retained Quarterly Earnings	< 0.0001
S&P 500	0.0020



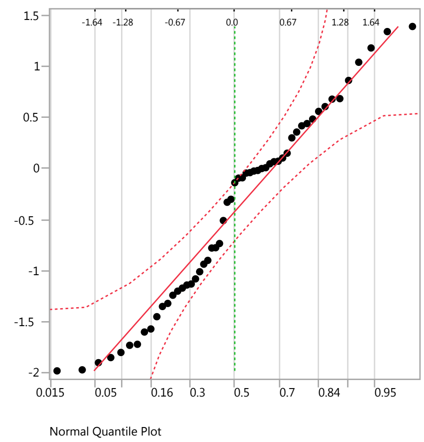
(a) GDP Per Capita



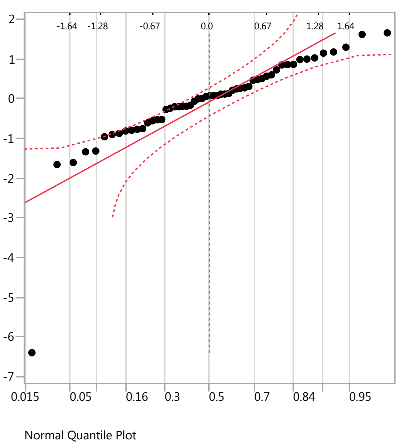
(b) As Reported Earnings



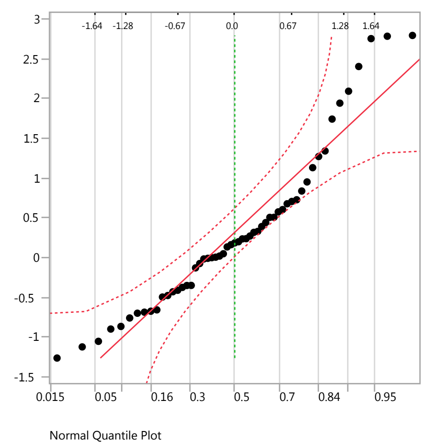
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

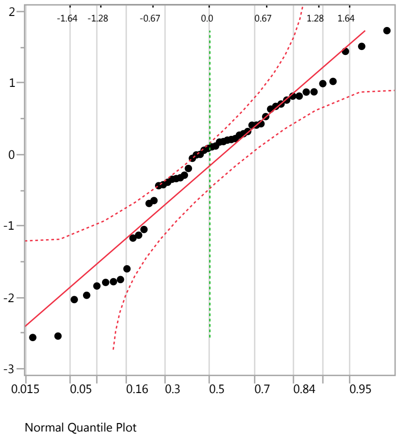
Figure 5.8: Normal probability plots of model residuals

Taking the analysis one step further, if the outlier from 4th quarter 2008 is excluded from the residual analysis, the normal probability plots improve and the goodness-of-fit measures increase substantially. Figure 5.9 shows how the residuals in the two earnings plots straighten along the diagonal line, and while some meandering is still present in the curves, their slopes conform better to the 45-degree angle without being influenced by the excluded leverage point.

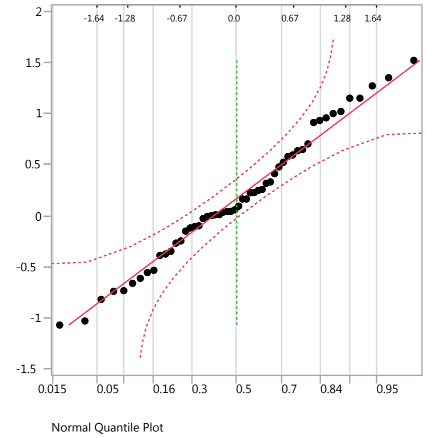
Not only do the normal probability plots behave better, their Shapiro-Wilk *p*-values improve by wide margins as shown in Table 5.39. With the outlier excluded, four of the five sets of component residuals are well above the 0.05 threshold, including those of the most volatile components, and the two that are below it, GDP Per Capita and S&P 500, are only slightly low.

Table 5.39: Goodness-of-Fit test results for residual analysis. 4th quarter 2008 data excluded

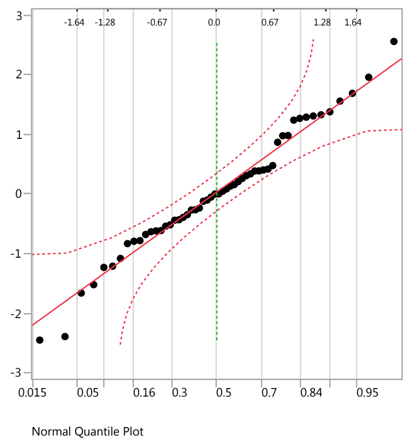
Shapiro-Wilks Test for Goodness-of-Fit	*P*-Value
GDP Per Capita	0.0122
As Reported Earnings	0.6320
Dividends	0.7048
Buybacks	0.0721
Retained Quarterly Earnings	0.8004
S&P 500	0.0027



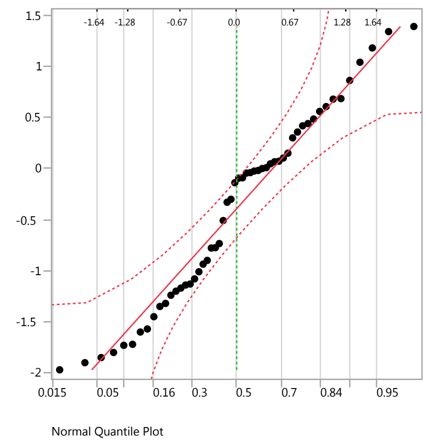
(a) GDP Per Capita



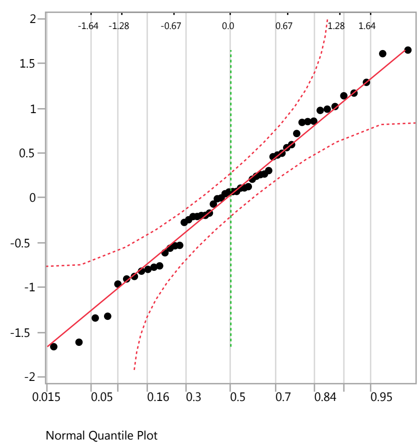
(b) As Reported Earnings



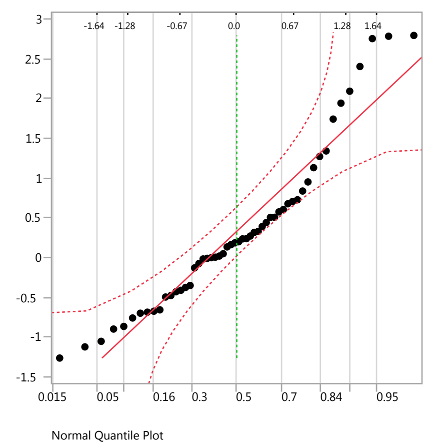
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.9: Normal probability plots of model residuals. 4th quarter 2008 data excluded

Removing the largest leverage point markedly improves the fit of the residuals, proving it to be an important outlier. While analysis of the data without the outliers is a useful and insightful exercise, removing them without cause is detrimental to the overall analysis effort and is not appropriate, even if it helps the model fit. Since it represents true system information for 4th quarter 2008, future analysis will keep the outlier, and all others.

Given these results, it is evident the model performs very well, predicting the actual data points with a fairly high degree of accuracy and erring in a random fashion. Considering the model also correctly replicates the trends in the data, it is apparent that it captures and simulates the dynamic behavior of the whole system and its inflection points, and it is suitable for use evaluating alternative scenarios.

5.7 Analysis of Hypothetical Scenarios

Each year the Federal Open Market Committee (FOMC) meets eight times, and as needed, to determine the target federal funds rate (Board of Governors of the Federal Reserve System, 2015b). As explained in Section 4.2, target interest rates are set in order to control inflation and unemployment, but many other aspects of the economy are impacted as well. Because the economy is a complex system, small changes in interest rates can have reverberating consequences across seemingly unrelated parts of the economy. For example, since the federal funds rate forms the basis for other lending rates across the country, changes by the Federal Reserve may directly cause bond yields to shift, real estate construction to slow or accelerate, and stocks may rise or fall depending on projected earnings and businesses' access to capital ("Interest Rate," 2016; Nielsen, 2016). Secondary and tertiary effects of a rate change may include an increase in stock buybacks or a slowdown in dividend growth due to businesses having access to more or less capital, which in turn also influences stock valuations.

More consequential perhaps are the time-lagged effects of rate increases on the economic system. As noted by Beachy, low interest rates in the early 2000's likely led to the housing boom of the mid-2000's (Beachy, 2012:10-12). In attempting to avoid a deflationary environment like that in Japan during the 1990's, the Federal Reserve rapidly dropped interest rates beginning in 2001 in order to bring the United States out of a recession. The additional credit available to consumers and businesses, combined with lowered lending standards and legislation aimed at helping more people to buy homes, allowed individuals and businesses to spend and expand beyond their normal limits, which quickly led to economic growth. As a result, the recession was minor on a historical scale and the Federal Reserve was credited as having prevented a deeper crisis (Ackman, 2001). Unfortunately, this rapid growth also likely fueled the subsequent housing bubble that gradually deflated from 2007 to 2012 with dramatic consequences (Taylor, 2009:1-4).

In hindsight, it is easy to recognize the Fed's mistake. Some leading economists have faulted the Federal Reserve for keeping interest rates at 50 year lows for so long in violation of the "Taylor Rule," which incorporates unemployment and inflation to determine the appropriate federal funds rate (Beachy, 2012:10-12; Taylor, 2009:1-4). Taylor even presented a "counterfactual" argument to bankers from the Federal Reserve in 2007, asserting that if the Taylor Rule had been followed in 2001 to 2006, the housing boom and bust would not have occurred (Taylor, 2009:4-6). This argument is summarized in the following two figures, the first of which was originally published in *The Economist*.

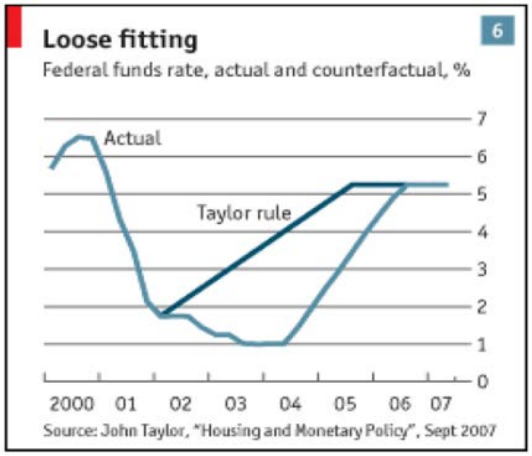


Figure 5.10: Depiction of actual interest rates and interest rates according to the Taylor Rule (Taylor, 2009:3)

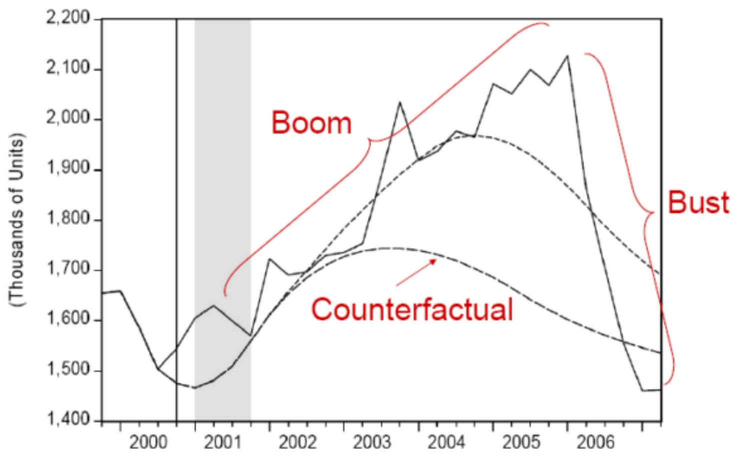


Figure 5.11: Taylor's counterfactual argument using autoregression to show how the housing boom and bust would have been avoided had the Taylor Rule been followed (Taylor, 2009:5)**.**

Questions about "what might have happened" if the Federal Reserve had acted differently often arise, and it is impossible to say precisely what might have occurred had the Federal Reserve pursued a different policy. Moreover, it is impossible to say how many crises have been averted due to well-timed monetary policy, or how many growth opportunities were stifled because of its overbearance, but hypothetical scenarios such as Taylor's can be used to illustrate alternative strategies and possible outcomes. Taylor used regression techniques to create his counterfactual argument, but because of the dynamical nature of the economy, it is also possible to model the economy as a complex adaptive system.

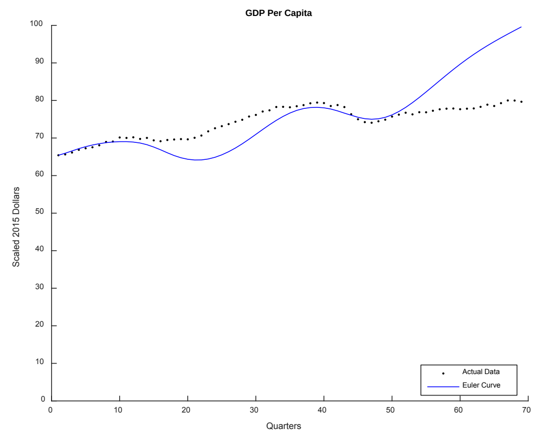
By changing the forcing function in the model created for this research, hypothetical scenarios can be tested and their short-term effects evaluated. An accurate model of a dynamic system can predict how the various components of the system will act in response to the actions of the other components, given a particular starting point. This view is necessarily limited of course, due to the presence of innumerable other factors that cannot be included in the model and that may exert themselves on the real system under different circumstances. For this reason, like in the case of regression analysis, there is danger in using a model to predict behavior over long periods of time. Moreover, as an adaptive system, the forcing functions introduced into the hypothetical scenarios are only notional and can never truly replicate the thoughts and feelings of decision makers in the real world.

Still, employing the model over short periods of time can provide significant insights, even creating a view of how the economy would have responded to different actions by the Federal Reserve in moments of crisis. While the Federal Reserve utilizes its own models to understand the effects of changes to interest rates, the model created in this research represents an additional tool that policy makers can use to investigate alternative situations. To do this, the model was used to examine three hypothetical scenarios, each featuring different potential Federal Reserve actions during the housing crash and economic recovery, or rather, the period of July 2007 to March 2015. This period is important not only because of its recent occurrence, but also because of the ongoing debate regarding the Federal Reserve's actions.

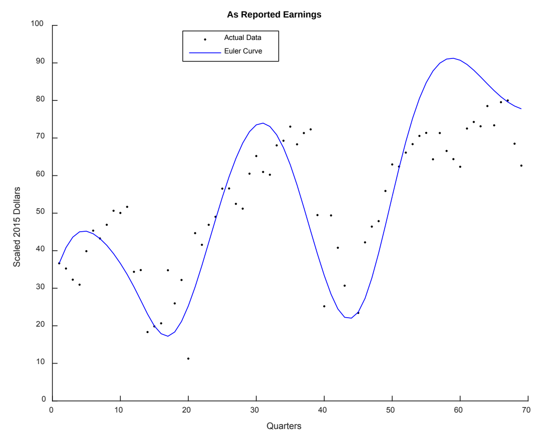
The housing market began its precipitous crash in 2007 and the Great Recession followed in 2008. The stock market also fell in late 2008, and many large American corporations and financial institutions were struggling or failing during this time period. By the beginning of 2009 the Federal Reserve had dropped interest rates to nearly zero and engaged in quantitative easing for several years thereafter. It was believed that increasing monetary liquidity would keep banks from failing, and easily accessible credit would allow businesses to keep functioning despite decreased consumer spending (Beachy, 2012). These were lessons economists learned in the many years since the Great Depression, and while they were widely accepted, many people feared that the huge increase in the money supply would lead to rampant inflation (Salsman, 2011). Others also raised concerns that, with interest rates already set to zero, the Federal Reserve had limited options if future interventions were necessary (Hilsenrath, 2015). Even some members of the Board of Governors of the Federal Reserve objected to the drastic measures taken (Fisher, Richard W., 2010). To date, this inflation has not materialized, but just as the Federal Reserve chose the path of near-zero interest rates, which seems to have led to soaring values in the stock market, it could have easily chosen an alternative strategy that was more tempered in its approach. The three scenarios in Sections 5.7.2 through 5.7.4 outline possible strategies the Federal Reserve might have taken, and the accompanying analysis gives the model results.

5.7.1 Developing a Baseline

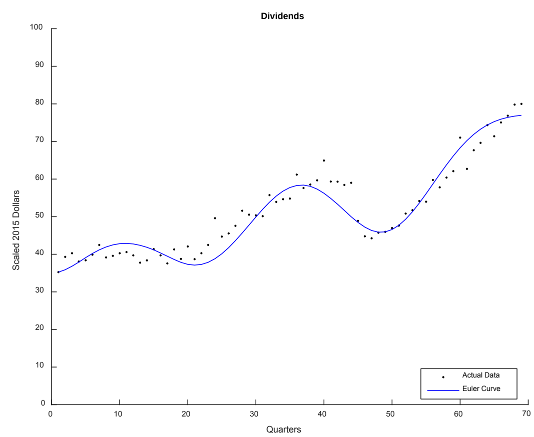
In order to fit the hypothetical scenario to the data, Euler's method was first used to establish the long-run trajectories of each of the model components. This was done by providing the model with a starting point in the first quarter of 1998, then using the derivatives calculated by the system of differential equations to extend the line quarter by quarter, as was previously explained in equations (4.11) through (4.12). The resulting curves in Figure 5.12 show how the system acts dynamically, as estimated by the model.



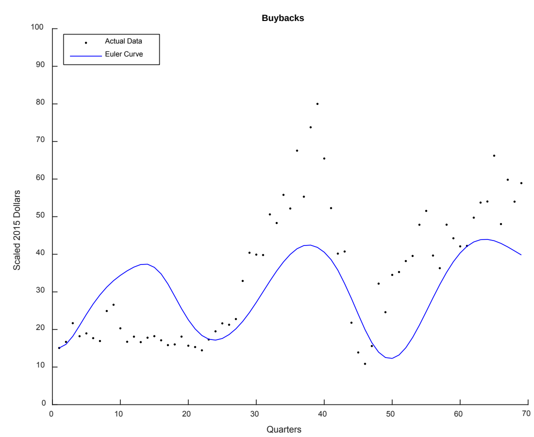
(a) GDP Per Capita



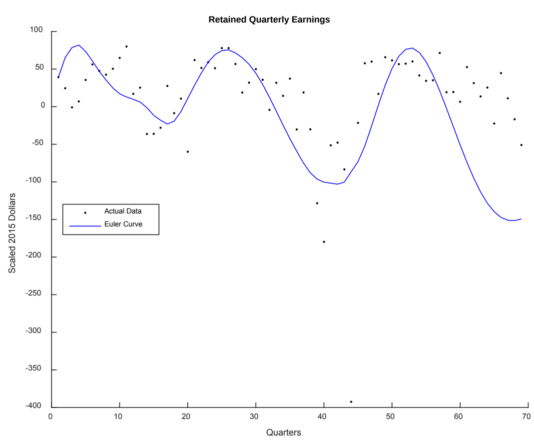
(b) As Reported Earnings



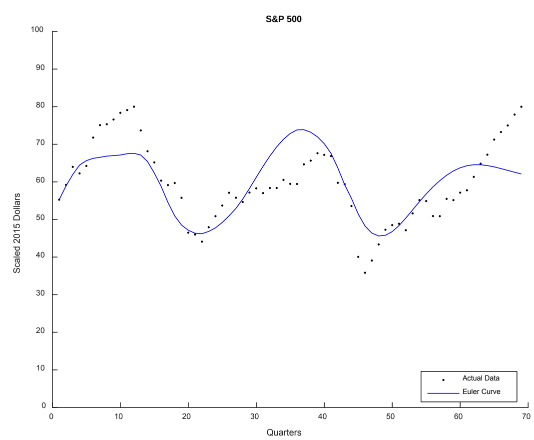
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.12: Predicted system behavior using Euler's method and a starting point in the 1st quarter of 1998, compared to actual system behavior

As expected, the artificial, generated curves differ substantially from the actual data, illustrating the limitations of the model. Nevertheless, the generated curves do capture most of the major trends in the data and predict all the major tipping points at nearly the correct times. Therefore, they are acceptable for simulating system behavior over the period of interest.

Interestingly, the onset of these tipping points is anticipated by the differential equations. Akin to the phenomena in options markets observed by Scheffer *et al.* and Bates, and discussed in Section 3.2, the differential equations exhibit a “slowing down” in the changes of the system states right before a bifurcation point is reached. Graphically, these tipping points are observable at the peaks and troughs of each curve. Quantitatively, a tipping point is attained when the first derivative of the curve is equal to zero. Prior to reaching zero however, the rate at which the first derivative changes also grows smaller and smaller, demonstrating that the second derivative of the curve is also decreasing, or rather, the rate of change of the differential equations is decreasing. This is apparent in Table 5.40 where the first column indicates the S&P 500 state for each period in time; the second column shows the derivative of this curve, the rate of change of the component state; and the third column shows the second derivative of the curve, or the rate of change of the changes in the curve. Since the table is composed of discrete quarterly data, not all the derivative values reach zero when they are expected to. If the data were continuous, the exact transition point of each curve and its derivatives could be ascertained.

Table 5.40: First and Second Derivatives of S&P 500

Quarter	Scaled S&P 500 Euler Values	First Derivative	Second Derivative
Jan-03	46.337	-0.791	0.603
Apr-03	46.204	-0.132	0.659
Jul-03	46.823	0.618	0.751
Oct-03	47.785	0.963	0.345
Jan-04	49.214	1.429	0.466
Apr-04	50.978	1.764	0.335
Jul-04	52.944	1.966	0.202
Oct-04	55.408	2.465	0.499
Jan-05	58.251	2.842	0.377
Apr-05	61.245	2.994	0.152
Jul-05	64.150	2.905	-0.089
Oct-05	66.881	2.730	-0.175
Jan-06	69.320	2.439	-0.291
Apr-06	71.336	2.016	-0.423
Jul-06	72.884	1.548	-0.468
Oct-06	73.837	0.952	-0.596
Jan-07	73.883	0.046	-0.906
Apr-07	73.213	-0.670	-0.716
Jul-07	71.983	-1.230	-0.560
Oct-07	70.168	-1.815	-0.585
Jan-08	67.566	-2.602	-0.788
Apr-08	63.691	-3.875	-1.272
Jul-08	59.252	-4.440	-0.565
Oct-08	55.607	-3.644	0.795
Jan-09	51.444	-4.163	-0.519
Apr-09	48.262	-3.182	0.982
Jul-09	46.384	-1.878	1.303
Oct-09	45.620	-0.764	1.114
Jan-10	45.797	0.176	0.940
Apr-10	46.767	0.970	0.794
Jul-10	48.381	1.614	0.644
Oct-10	50.367	1.986	0.372
Jan-11	52.550	2.183	0.197
Apr-11	54.736	2.186	0.003
Jul-11	56.789	2.053	-0.133
Oct-11	58.678	1.889	-0.164

The quarterly data leading up to January 2007 and October 2009 in Table 5.40 exhibit how the derivatives indicate the approach of a critical turning point in the S&P 500 curve. Specifically, the first derivative of the S&P 500 curve grows smaller as the data begins to peak. Accordingly, the second derivative decreases to a low of -0.906 until the tipping point is attained, at which point it again begins to increase. This effect is shown graphically in Figure 5.13.

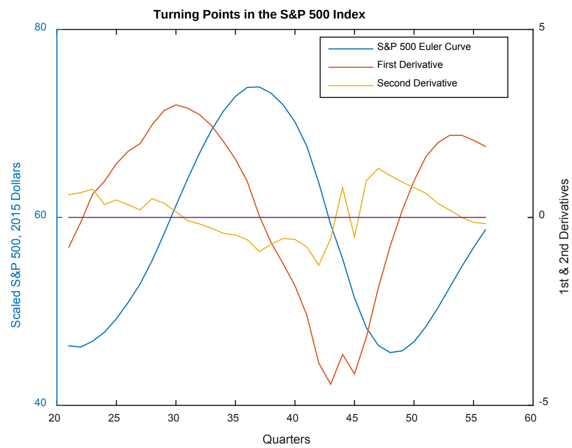


Figure 5.13: Turning points in the S&P 500 Index as shown by the first and second derivatives

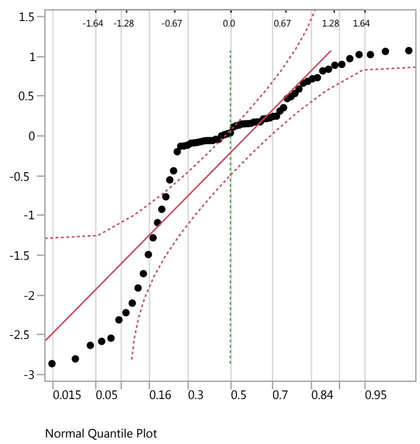
The accuracy of the Euler curves is further illustrated through hypothesis testing. Using the methodology introduced in Section 4.6.3, a *t*-test assuming equal variances was conducted on the data generated by the Euler curves and the actual data. *P*-values less than 0.05 indicate the two data sets are significantly different, but *p*-values above 0.05 results in a failure to reject the null hypothesis that the two sets are the same. As seen in Table 5.41, only the Euler curves for Buybacks and Retained Quarterly Earnings differ markedly from the actual data according to the *t*-test results, and only Buybacks results in a rejection of the null hypothesis, primarily due to the inconsistent nature of this data relative to the model. Whereas the model predicts buybacks to increase when earnings and stocks increase, this does not always happen in the data, illustrating one shortcoming of the model. These results substantiate the accuracy of the Euler curves as predictors of the actual system and support their use as a baseline for analyzing the hypothetical scenarios in the next three sections.

Table 5.41: Hypothesis test results for *t*-test on predicted Euler curves vs. actual data

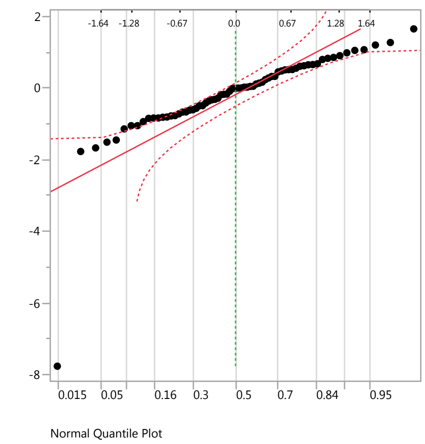
	P-Value	Reject at $\alpha = 0.05$?
GDP Per Capita	0.2961	No
As Reported Earnings	0.5335	No
Dividends	0.7664	No
Buybacks	0.0392	Yes
Retained Quarterly Earnings	0.0755	No
S&P 500	0.9758	No

A residual analysis of the generated Euler curves indicates that, while the independently-flowing model does contain large errors, the distribution of these errors remains largely normal compared to the fitted curves. The major deviations in this case reflect the instances where the predicted curves depart from the actual data, including the leverage point outlier from the 4th quarter 2008. However, these deviations are relatively minor considering the model received only a single vector input of real data in the 1st quarter of 1998.

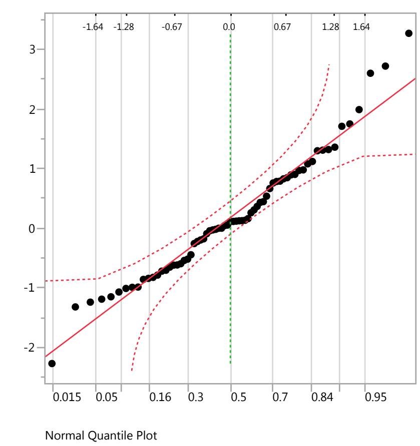
The normal probability plots shown in Figure 5.14 illustrate that most of the residuals are located inside the lines marking the 95% confidence interval, and while a few of the plots exhibit a meandering pattern about the normal line, only GDP Per Capita is grievously non-normal. This poor behavior in the GDP plot is expected however, given the dynamic system predicts all the components will naturally rise and fall and not progress in a straight-line fashion as GDP does. Besides, the reason for the straight-line trend of GDP, in contrast to the fluctuating behavior of the other components, is attributable to the macro-scope of this variable. As discussed in Section 5.3.1, GDP is measured on a macro-scale, whereas the other model variables are collected on the more micro-scale of the S&P 500. Thus, poor behavior in the model residuals for GDP is not considered a critical shortfall. In all, the externally studentized residuals of the Euler curves exhibit surprisingly normal behavior considering the model data is completely and independently predicted over the period of 17 years.



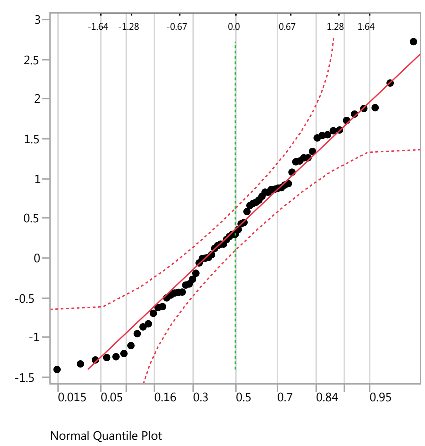
(a) GDP Per Capita



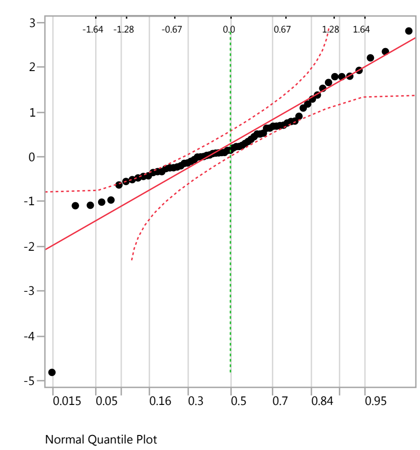
(b) As Reported Earnings



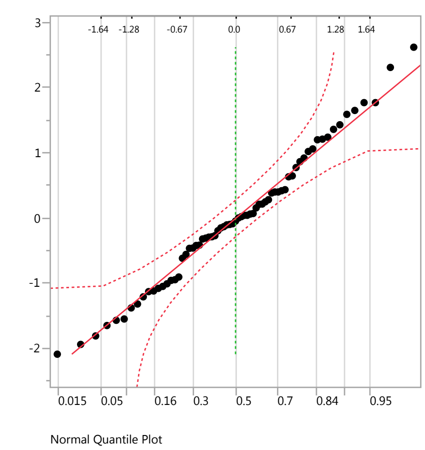
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.14: Normal probability plots depicting residual analysis of predicted Euler curves

The Shapiro-Wilk Goodness-of-Fit Test was also conducted on these residuals. Somewhat like the fitted curves, where the Shapiro-Wilk Test contributed to the conclusion that the residuals were normally distributed, the test on the predicted Euler curves indicates that several of the residual plots are distributed in a normal fashion. The *p*-values in Table 5.42 indicate with significant certainty that Dividends, Buybacks, and S&P 500 residuals are normally distributed at an alpha value of 0.05. Yet, even where the test suggests non-normality, as noted earlier, the Shapiro-Wilk Test is a conservative assessment of normality and does not necessarily indicate model inadequacy. Since five of the six normal probability plots do not contain extreme departures from the normal line, and the predicted curves appear to correctly replicate the general trends in the actual data, it is reasonable to conclude that the fitted Euler curves will serve as adequate baselines for further analysis and comparison in the three hypothetical scenarios that follow.

Table 5.42: Goodness-of-Fit test results for residual analysis of the predicted Euler curves

Shapiro-Wilks Test for Goodness-of-Fit	*P*-Value
GDP Per Capita	< 0.0001
As Reported Earnings	< 0.0001
Dividends	0.1064
Buybacks	0.4376
Retained Quarterly Earnings	< 0.0001
S&P 500	0.6741

5.7.2 Scenario 1: Gradually Increasing Interest Rates After 1st Quarter 2009

The first scenario examines how the modeled economic system would have reacted if the Federal Reserve had followed an approximate version of the Taylor Rule beginning in 2008. That is, after quickly dropping interest rates to almost zero by the end of 2008, the Federal Reserve would have begun slowly and steadily increasing rates until March 2015. Naturally, such a course of action would have precluded the strategy of quantitative easing and may have caused deflation and a deeper recession, but it also would have allayed fears of high future inflation and would have given the Federal Reserve more options in regards to lowering the interest rate again if needed. Indeed, such a strategy would have been more in line with the desires of the interest rate "hawks" that argued for higher interest rates following the onset of the recession. Figure 5.15 graphically depicts the actual federal funds rate and the hypothetical federal funds rate used for Scenario 1 over the period of January 1998 to March 2015. The departure of the hypothetical rate from the actual interest rate is apparent beginning in April 2009 and increases 0.15 points each quarter until March 2015.

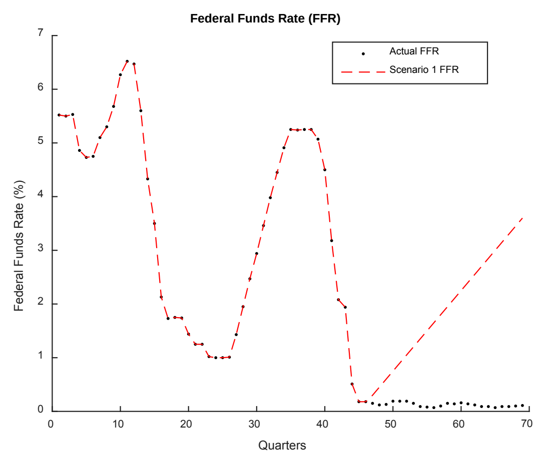


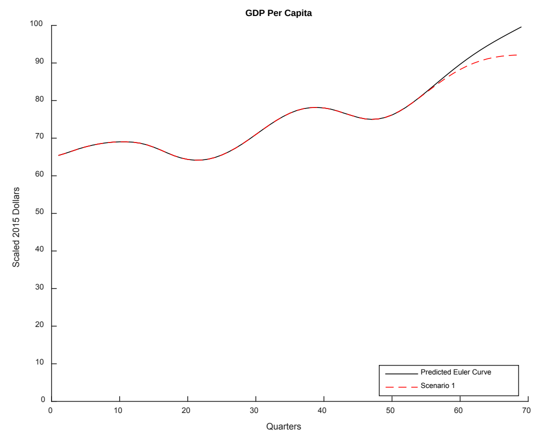
Figure 5.15: Federal funds rate for hypothetical Scenario 1

Following the interest rate strategy outlined in Scenario 1 would have produced mixed results. According to the model, GDP Per Capita would not have climbed as quickly after the 2008 recession, ending the test period 7.5% lower than in the baseline case. While this growth is slower than the baseline, the model does not show more negative growth that would be indicative of a further, or deeper, recession. Likewise, the model predicts that the As Reported Earnings of businesses in the S&P 500 index would have suffered as a result of the more restricted capital from the higher interest rates. Compared to the baseline case, As Reported Earnings ended the test period $87 billion lower. The lower earnings seem to carry over to dividend payments, which also end lower at $12 billion.

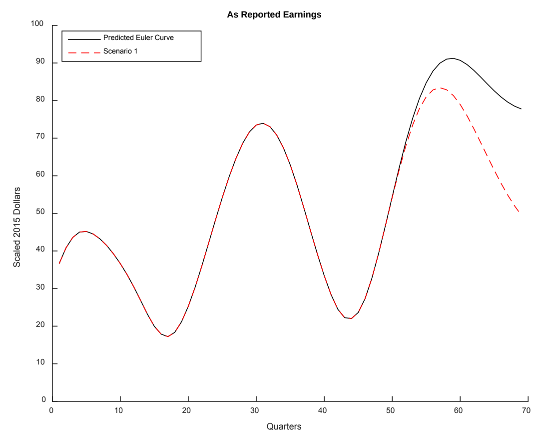
Interestingly however, the higher interest rates seem to have little effect on the S&P 500, which ends the period only slightly below the baseline. The S&P 500 price level may have been helped by the $16 billion increase in buybacks at the end of the test period. Retained Quarterly Earnings also increases, as would be expected when interest rates go up. Since capital is more expensive to obtain, businesses may be more likely to retain current earnings to facilitate future growth. Moreover, the increase in retained quarterly earnings seems to relate to the decrease in dividend payments. The plotted results for Scenario 1 are visible in Figure 5.16, and the quantitative difference at the end of the period is detailed in the following table.

Table 5.43: Scenario 1 Results at end of test period

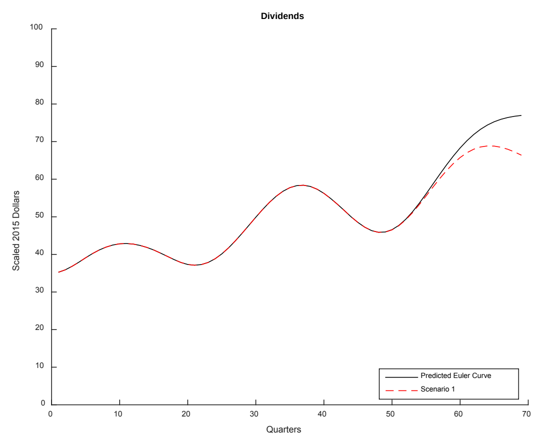
	Baseline	Scenario 1	Difference
GDP Per Capita ($)	69,255.58	64,054.40	-5,201.17
As Reported Earnings (Bil $)	240.6	153.2	-87.4
Dividends (Bil $)	90.4	78.0	-12.4
Buybacks (Bil $)	98.0	114.1	16.1
Retained Quarterly Earnings (Bil $)	-132.4	-80.7	51.7
S&P 500 Index (Price Level, $)	1,579.14	1,566.31	-12.83



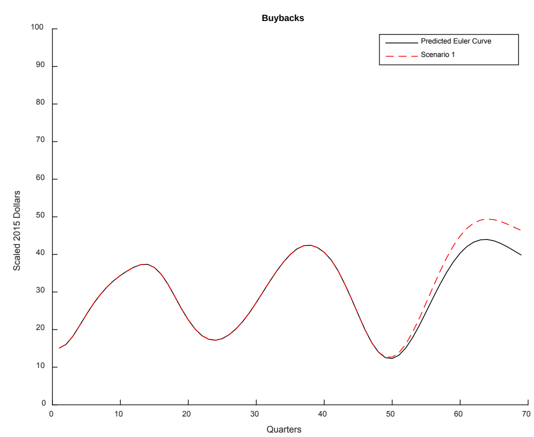
(a) GDP Per Capita



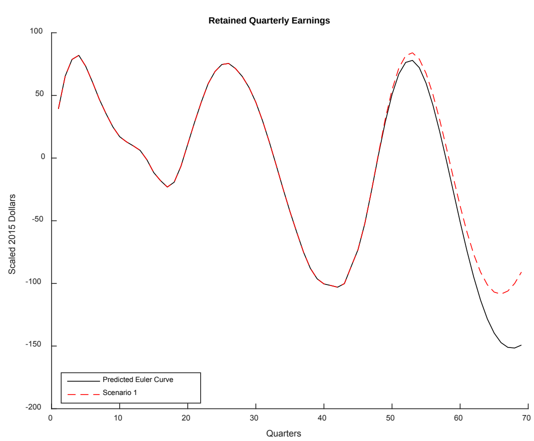
(b) As Reported Earnings



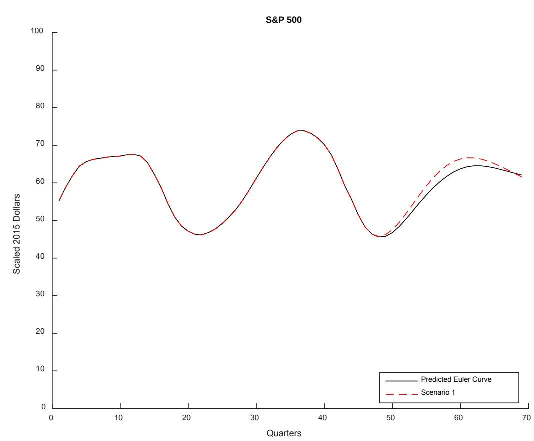
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.16: Plots comparing Scenario 1 results to the baseline case

Building on the previous assumption that the generated data is sufficiently normal and adequately well-behaved, a simple *t*-test is used to test the differences between the baseline case and the results of Scenario 1. This test shows the change in the Federal Funds Rate does cause significantly different results for each of the model components. Using an alpha value of 0.05, the null hypothesis states that the mean of the baseline data is the same as the mean obtained from Scenario 1. The alternative hypothesis states the opposite, that the means are not equal to one another.

$$H_0 : \mu^{(j)}_{Baseline} = \mu^{(j)}_{Scenario\,1}$$
$$H_1 : \mu^{(j)}_{Baseline} \neq \mu^{(j)}_{Scenario\,1}$$

The hypotheses are tested using the equation

$$t = \frac{\mu^{(j)}_{Baseline} - \mu^{(j)}_{Scenario1}}{\sigma^{(j)}_{Baseline} / \sqrt{n}}, \tag{5.5}$$

where σ represents the standard deviation and n is the sample size, which is then used to calculate the following *p*-values for each of the model components.

Table 5.44: Scenario 1 *T*-test results

	***P*-value**	**Reject at $\alpha = 0.05$?**
GDP Per Capita	0.002028	Yes
As Reported Earnings	0.000215	Yes
Dividends	0.000934	Yes
Buybacks	0.000023	Yes
Retained Quarterly Earnings	0.000321	Yes
S&P 500	0.000038	Yes

The *p*-values in Table 5.44 indicate that all the model components are affected by the change in the federal funds rate. This observation is consistent with the plots in Figure 5.16 that show large changes compared to the baseline, except for the S&P 500, which seems to stay close the baseline curve.

Alternatively, prediction intervals around the end values of Scenario 1 are used to show whether the generated curves for Scenario 1 are significantly different than the curves for the baseline case. Again assuming the data is normally distributed, and using an alpha value of 0.05, the 95% prediction intervals are calculated using equations (4.35)to (4.38) from Section 4.6.3, and repeated here as equation (5.6).

$$\left[\hat{x}^{(i)}_{n_{Scenario\,1}} \pm t_{n,\alpha/2} s^{(i)}_{Scenario\,1} \sqrt{1+\text{Distance value}}\right], i = 1, 2, \ldots, 6 \tag{5.6}$$

$$\begin{aligned}&\alpha = 0.05\\&n = 69\\&t_{n,\alpha/2} = 1.995\\&\text{Distance Value} = 0.05673\end{aligned}$$

This method indicates that all of the model components except S&P 500 are significantly impacted by the change in interest rates in this scenario. Five of the six variables have statistically significantly different end states than the baseline case, showing that interest rates have a large effect on the dynamics of the system over a five year period. The predicted S&P 500 cannot be said to be statistically different however, as the baseline overlaps with the 95% prediction interval. Table 5.45 details these results, which largely coincide with the findings of the hypothesis test, as expected.

Table 5.45: Prediction intervals on Scenario 1 end of period value

	95% Prediction Interval	Baseline	Overlap with Baseline?
GDP Per Capita ($)	[61590.93, 66517.87]	69255.58	No
As Reported Earnings (Bil $)	[101.49, 204.83]	240.56	No
Dividends (Bil $)	[71.73, 84.24]	90.42	No
Buybacks (Bil $)	[102.29, 125.92]	98.00	No
Retained Quarterly Earnings (Bil $)	[-105.88, -55.56]	-132.40	No
S&P 500 Index (Price Level, $)	[1512.15, 1620.46]	1579.14	Yes

Given these results, the model indicates that steadily raising interest rates according to Scenario 1 would have statistically significantly diminished the performance of the S&P 500 system over the long term, even if the S&P 500 itself recovered from the downturn. It is likely that GDP growth would have been slower with higher interest rates, and business earnings would have been much lower. These conclusions are generally in line with conventional interest rate models utilized by economists.

5.7.3 Scenario 2: Higher Interest Rates During Great Recession

The second scenario considers how the model would have reacted if the Federal Reserve had pursued a less drastic, more conservative attitude toward dropping interest rates. In seeking to avoid quantitative easing while keeping open the possibility of future rate drops, the FOMC could have chosen to keep rates higher in 2008, waiting to see how the economy would react to its first rate drops of nearly 3 points. Like the first scenario, such a course of action would have avoided quantitative easing and the monetization of U.S. debt. However, such an action may have risked deflation and a deeper recession. Figure 5.17 illustrates the federal funds rate used for Scenario 2. The departure of the hypothetical rate from the actual interest rate is apparent beginning in July 2008 and remains steady until March 2015.

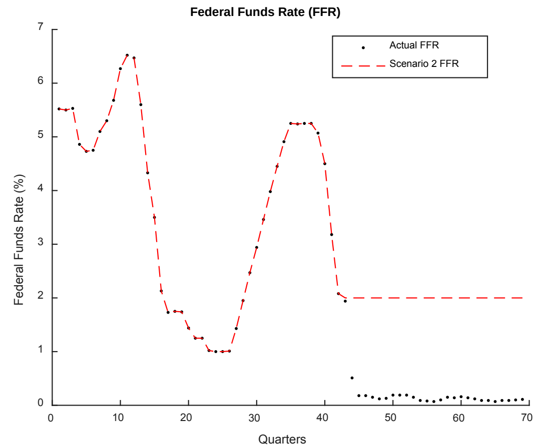
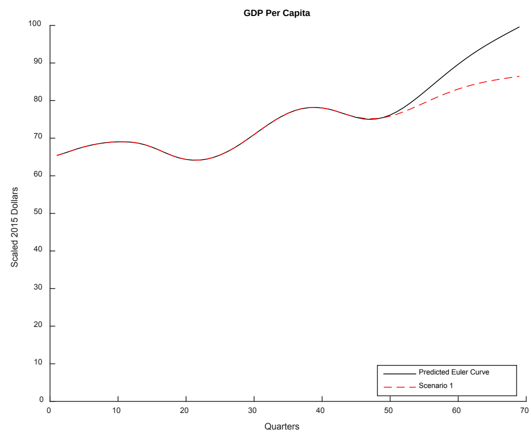


Figure 5.17: Federal funds rate for hypothetical Scenario 2

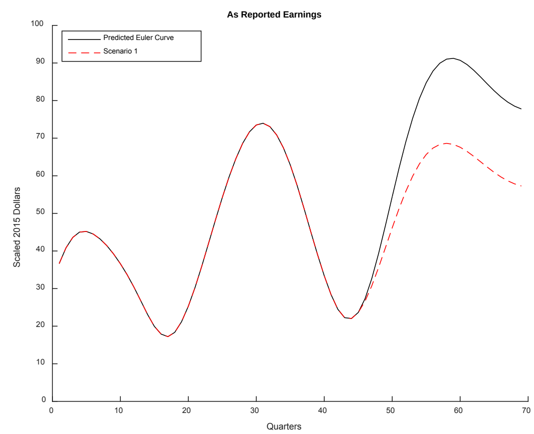
The resulting end states for Scenario 2 are not dissimilar to the results for Scenario 1. In this case however, because the interest rate was never dropped to zero, GDP recovers much more slowly than in the baseline case. The S&P 500 also ends at a much lower level even though As Reported Earnings are higher. Dividends and Buybacks are somewhat lower than in Scenario 1, but are still low compared to the baseline. Here, it seems that higher interest rates have a dampening effect on the model components, perhaps because the decreased liquidity tends to restrict the volatility of the six variables. The quantitative results for this scenario are shown in Table 5.46, and the scaled results are plotted in Figure 5.18.

Table 5.46: Scenario 2 results at end of test period

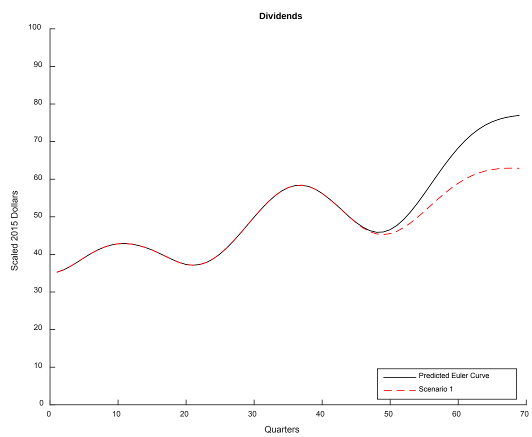
	Baseline	Scenario 2	Difference
GDP Per Capita ($)	69,255.58	60,111.85	-9,143.73
As Reported Earnings (Bil $)	240.6	177.2	-63.4
Dividends (Bil $)	90.4	73.9	-16.5
Buybacks (Bil $)	98.0	108.5	10.5
Retained Quarterly Earnings (Bil $)	-132.4	-54.9	77.5
S&P 500 Index (Price Level, $)	1,579.14	1,446.54	-132.60



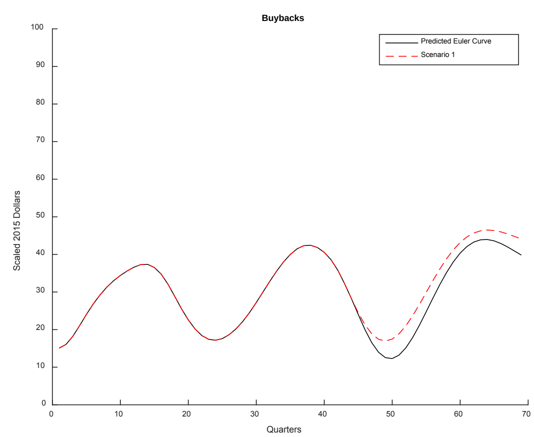
(a) GDP Per Capita



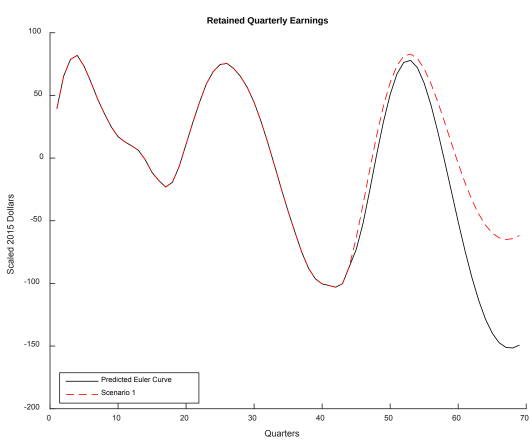
(b) As Reported Earnings



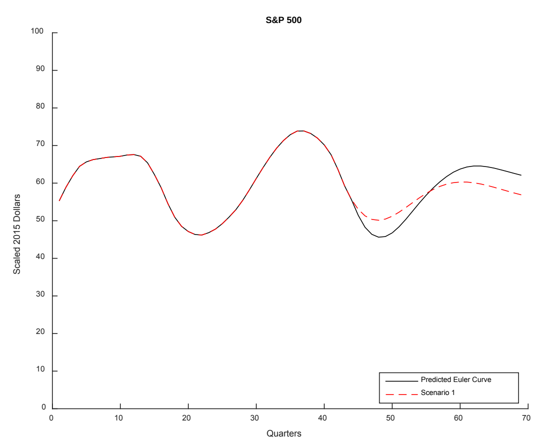
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.18: Plots comparing Scenario 2 results to the baseline case

Performing a hypothesis test on the results for Scenario 2 versus the baseline shows that holding the interest rate at 2% for the duration of the recession and recovery has a significant effect on the model components. According to the *p*-values, only the S&P 500 would not have been significantly influenced by the change at an alpha of 0.05, which may only testify to the fact that the stock market would have recovered from its 2008-2009 crash because of the improving economy in general, regardless of minor changes to the federal funds rate. The other components, especially the most volatile ones like Buybacks and As Reported Earnings, seem to be the most affected by the change, which corroborates expectations of these variables as being responsive to interest rates because of their ability to make extra capital available to consumers and common stock companies. Table 5.47 summarizes these findings.

Table 5.47: Scenario 2 *T*-test results

	P-value	Reject at $\alpha = 0.05$?
GDP Per Capita	0.000088	Yes
As Reported Earnings	0.000001	Yes
Dividends	0.000024	Yes
Buybacks	0.000000	Yes
Retained Quarterly Earnings	0.000041	Yes
S&P 500	0.284676	No

The 95% prediction intervals for Scenario 2 yield slightly different results than those found in the hypothesis test, as shown in Table 5.48. The prediction intervals consider only the end state of the system as a result of the changed interest rate. Here, four of the six model components are predicted to be statistically different from the baseline case, including the S&P 500 which ends 132 points lower, illustrating a reduced recovery for the stock market as a result of the more stringent monetary policy. As discussed in regards to the hypothesis testing, these results are entirely plausible and not surprising. The model seems to accurately capture the inherent dynamics between the model components and the exogenous forcing function.

Table 5.48: **Prediction intervals on Scenario 2 end of period value**

	95% Prediction Interval	Baseline	Overlap with Baseline?
GDP Per Capita ($)	[54319.08, 65904.61]	69255.58	No
As Reported Earnings (Bil $)	[108.78, 245.60]	240.56	Yes
Dividends (Bil $)	[61.73, 86.07]	90.42	No
Buybacks (Bil $)	[96.66, 120.36]	98.00	Yes
Retained Quarterly Earnings (Bil $)	[-108.53, -1.24]	-132.40	No
S&P 500 Index (Price Level, $)	[1326.69, 1566.37]	1579.14	No

5.7.4 Scenario 3: Gradual Reduction of Interest Rates Followed by Gradually Increasing Rates

The final scenario imagines how the model components would have acted if the Federal Reserve had decreased interest rates more gradually then increased them steadily from July 2007 to March 2015. This scenario represents a very cautious reaction to the recession, and violates standard economic norms regarding interest rates and recessions, but it is illustrative of the effect that interest rates have on the model (Beachy, 2012:11). Figure 5.19 depicts the actual federal funds rate and the hypothetical rate for scenario 3. The departure of the hypothetical rate from the actual interest rate is apparent beginning in July 2007 and decreases 0.25 points each quarter until July 2010. It then remains constant from July 2010 to July 2012, before increasing by 0.25 points per quarter until March 2015.

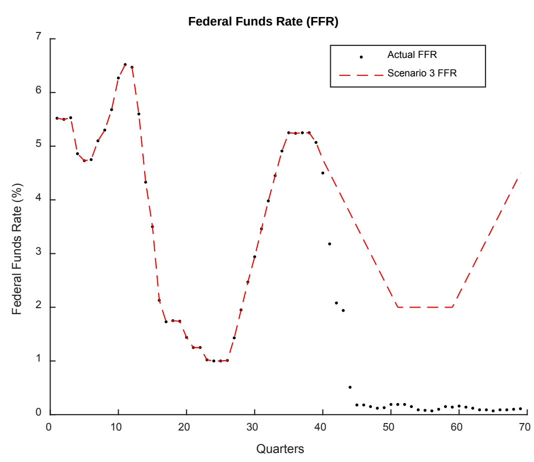
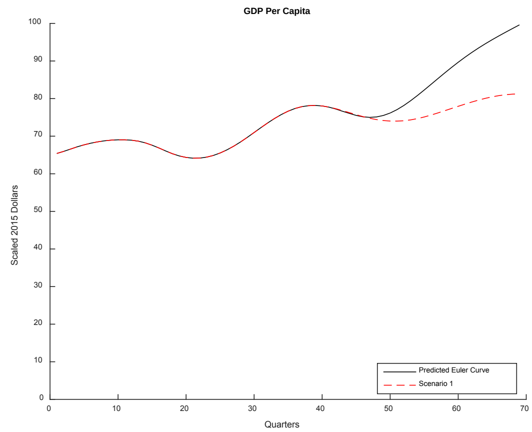


Figure 5.19: Federal funds rate for hypothetical Scenario 3

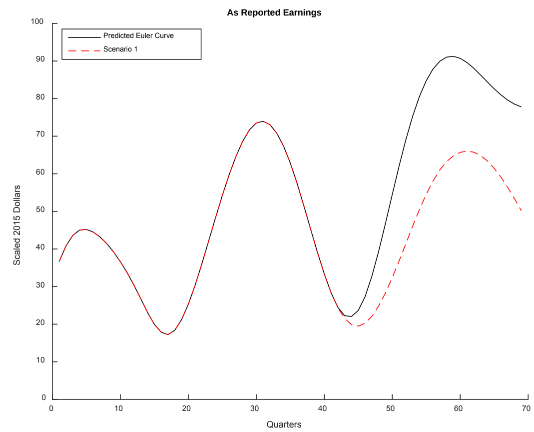
The dates and interest rate levels for Scenario 3 were chosen arbitrarily in order to represent a highly conservative approach to lowering interest rates. It also produces the lowest end results of the three scenarios. Table 5.49 shows that GDP Per Capita would have ended the test period nearly $13,000 lower than in the baseline case, a result that is similar to the actual data. Although the slow growth did not result in another recession in the model, if the model was continued forward over several years, it is suggested that a recession would have occurred very soon. Similarly, As Reported Earnings ends the test period substantially lower than the baseline, and the S&P 500 is down slightly. Characteristically, Retained Quarterly Earnings is much higher in this scenario, owing to the decreased access that businesses have to borrowed capital. In this case, Buybacks increased, possibly due to the upward trending S&P 500 late in the test period. This is reflective of reality in which corporate managers and boards may try to push stock prices higher when they are already high, and the behavior generated by the model seems consistent across all three scenarios and throughout the model in general. The plots in Figure 5.20 depict these finding graphically.

Table 5.49: Scenario 3 results at end of test period

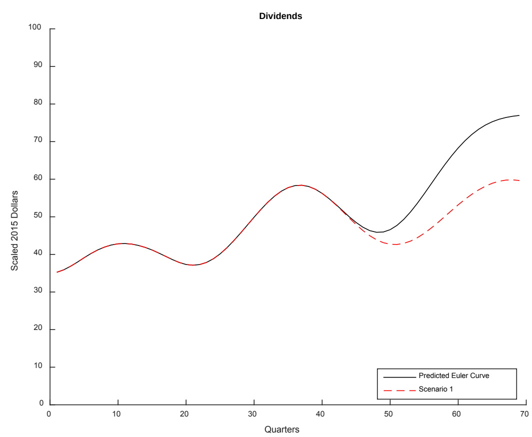
	Baseline	Scenario 3	Difference
GDP Per Capita ($)	69,255.58	56,484.68	-12,770.90
As Reported Earnings (Bil $)	240.6	155.2	-85.4
Dividends (Bil $)	90.4	70.1	-20.3
Buybacks (Bil $)	98.0	125.4	27.4
Retained Quarterly Earnings (Bil $)	-132.4	-40.4	92.0
S&P 500 Index (Price Level, $)	1,579.14	1,556.78	-22.36



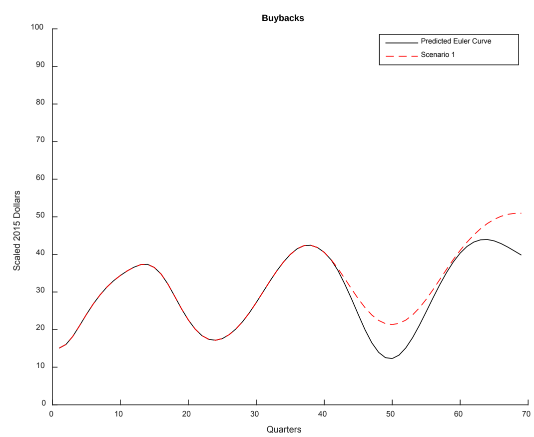
(a) GDP Per Capita



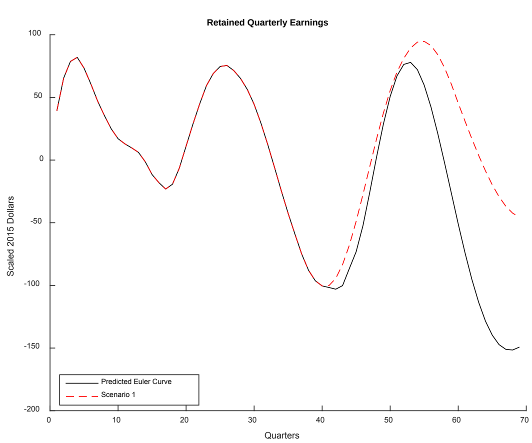
(b) As Reported Earnings



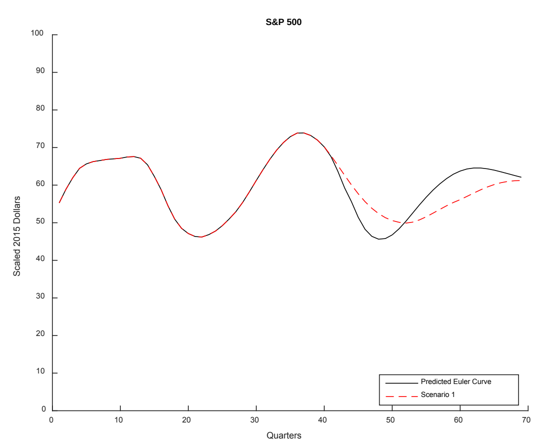
(c) Dividends



(d) Buybacks



(e) Retained Quarterly Earnings



(f) S&P 500

Figure 5.20: Plots comparing Scenario 3 results to the baseline case

The forcing function values in Scenario 3 are very different from those in the baseline model, so it is not surprising that the hypothesis test reveals these differences. Like in the previous two scenarios, the *p*-values here indicate that Scenario 3 is significantly different from the baseline case in at least five of the six model components. Only the S&P 500 is shown to be similar in both circumstances, which is anticipated based on the behavior of this component in the other scenarios. This outcome is observable in the plotted data and also in Table 5.50. In all the hypothetical scenarios, the modeled S&P 500 seems resilient to small changes in the forcing function.

Table 5.50: Scenario 3 *T*-test results

	***P*-value**	**Reject at $\alpha = 0.05$?**
GDP Per Capita	0.000014	Yes
As Reported Earnings	0.000000	Yes
Dividends	0.000003	Yes
Buybacks	0.000002	Yes
Retained Quarterly Earnings	0.000010	Yes
S&P 500	0.214771	No

Finally, the prediction intervals for Scenario 3 reinforce the findings visible in the plotted data and in the hypothesis data. Summarized in Table 5.51, if interest rates had been kept higher going into the 2008 recession, the end state of GDP Per Capita, Dividends, Buybacks, and Retained Quarterly Earnings would have been very different. Only the S&P 500 and As Reported Earnings seem to act more independently of rate increases, and the end results of these variables in Scenario 3 would have been very close to that of the baseline within the period tested. Overall, the interest rate policy in Scenario 3 seems to drastically affect the economic system described by the model.

Table 5.51: Prediction intervals on Scenario 3 end of period value

	95% Prediction Interval	Baseline	Overlap with Baseline?
GDP Per Capita ($)	[47417.56, 65551.79]	69255.58	No
As Reported Earnings (Bil $)	[65.06, 245.33]	240.56	Yes
Dividends (Bil $)	[52.42, 87.74]	90.42	No
Buybacks (Bil $)	[106.46, 144.35]	98.00	No
Retained Quarterly Earnings (Bil $)	[-125.01, 44.23]	-132.40	No
S&P 500 Index (Price Level, $)	[1383.55, 1730.00]	1579.14	Yes

5.8 Summary

This chapter showed how the model was implemented. After processing data for the seven model components, the method of least squares was used to fit the model to the actual data. Several strategies were used to fit this data and determine the coefficients of the system of differential equations, but ultimately, fitting the weighted model to the data points resulted in a tighter fitting model than could be obtained by fitting the unweighted model or basing the fit on the slopes or secants. Further, residual analysis of the fitted, weighted model proved that it was sufficiently normal to employ some traditional statistical tests such as hypothesis tests and prediction intervals.

While subject to limitations, the model attempts to capture the dynamic economic system well enough to analyze certain alternative scenarios. While purely hypothetical, and potentially contrary to conventional economic theory, the three scenarios were used to demonstrate the utility of the model rather than prescribe a path that the Federal Reserve should have taken. To this end, the model performed well, demonstrating that inherently dynamic economic systems can be modeled as complex adaptive systems; and that such a model, formulated using a system of differential equations can be used to imagine and simulate alternative realities based on decision maker actions.

VI. Conclusions and Recommendations

6.1 Chapter Overview

After introducing the research objective and examining existing literature, this thesis explored the subject of economic dynamics in a context relevant to national defense. It then described the creation of a new model and methodology for analyzing economic systems, and tested the model in Chapter 5. This chapter will summarize the conclusions and contributions of the research, and suggest some ideas for future study.

6.2 Summary of Findings

The objective of this pilot research was to show that an economic system can be modeled as, and characterized by, a complex adaptive system, using a system of differential equations to solve an inverse problem. This was done to build a foundation on which new tools and methodologies could be developed to assist national policy makers' understanding of economic dynamics and financial markets.

As world markets and finances become more globalized and interconnected this understanding is increasingly important. History provides examples of private investors who have taken advantage of market imbalances to the detriment of large and developed countries. In addition, foreign military thinkers have proposed ways in which financial warfare may become a dominant front in future conflicts. However, U.S. military doctrine is segmented in this regard and does not typically consider economic factors as part of a comprehensive and multifaceted strategy.

Moreover, current economic theory does not thoroughly acknowledge the dynamic relationships between the various aspects of an economy, especially in the time dimension.

Instead, much of economic theory is based on constrained optimization that examines relationships at a single point in time. When time is considered, economic indicators are predicted or forecasted using autoregression, which generally fails to recognize the feedback processes and critical bifurcation points that naturally occur in complex systems.

Rather, economic systems should be viewed as networks whose components influence each other over time in dynamical and adaptive ways. Casting the components of an economy in this light is more realistic and permits construction of the model introduced in Chapter IV. This model is based on a system of differential equations whose coefficients determine the strength of relationship between model variables. To account for the turning points, or peaks and troughs that characterize an economy's fluctuations over time, the model employs a functional form based on logistic differential equations for populations. This is important because the equations include carrying capacities that limit the growth or decline of any particular variable in the model based on the states of the other variables. The structure of the system of differential equations and the model components' interactions with one another are key features in the model for simulating real world economic behavior.

Because the coefficients of the model are initially unknown and always specific to the current problem, an inverse problem was solved to determine the values of these coefficients. The process for solving the problem is based principally on the method of least squares, with nonlinear optimization employed to minimize the sum of squares. After the model was fit with the determined coefficients, residual analysis was implemented to assess model adequacy and common statistical tools such as hypothesis testing and prediction intervals were used to validate the model fit.

Three approaches were utilized to fit the model. The first minimized the sums of squared errors between the actual data and the predicted data points. This method ultimately resulted in the best-fitting model, with all of the model components achieving R-squared values above 0.5 and two components scoring above 0.9. Mean squared errors also tended to be the smallest under this approach, with the most extreme real data point of Buybacks falling within the prediction interval of the modeled data. Because of its overall superior performance, the resulting model from this fitting approach was used as a baseline for the hypothetical scenarios.

Since the data for this test is highly volatile, the model was also fit to the slopes between data points and the secants between data points. It was anticipated that one of these approaches might better match the trends in the actual data. At times, upon visual inspection of the plotted data curves, this proved to be the case, especially with the secants, but often these fits exaggerated the movements of the true data or produced validation curves that moved opposite to the actual data. Furthermore, while minimizing the squared error between the actual slopes and secants and the predicted slopes and secants sometimes produced admirable fits with low mean squared errors, the R-squared values were typically quite low, scoring below 0.5 on average, and these models were rejected in favor of the model based on the data.

Euler's Forward Method was introduced in the methodology and was used to create a baseline for the evaluation of alternative scenarios. Applying this method with the coefficients from the weighted data model produced curves that were similar to the actual data. This is encouraging because the Euler method takes only a single vector of actual data as input at the beginning of the test period, then projects system behavior forward using only the results of its own differential equations to predict future system states. Although errors were present between the Euler curves and the actual data, the model accurately portrayed long-run system behavior over a period of 17 years while correctly predicting every major turning point in the data for the six variables. Hypothesis testing further reinforced these results, showing that five of the six modeled curves were not statistically different from their real-life counterparts. Having established the Euler curves as a suitable proxy for actual system performance, these curves functioned as a baseline for testing the influence of interest rates on the system of components in three distinct scenarios.

The federal funds rate is a central ingredient in the model, and is essentially the driver of the system as a complex adaptive system. Because the federal funds rate is exogenous to the model and independently controlled by the Federal Reserve, it was used in the hypothetical scenarios to inject new conditions to which the system must adapt. The three scenarios were based on contrived monetary policies that the Federal Reserve could have implemented during and after the 2008 recession. In each of these cases, the modeled system adjusted to the new interest rate environment, resulting in curves that proved statistically different from the baseline, and with end state values and 95% prediction intervals positioned well-away from the baseline end states.

These results establish the utility of the model both as a forward predictor of the system components and as a tool for retroactively studying system behavior under alternative conditions. While still in the early stages of development, the model serves as a foundation for future research and as a tool for policy makers interested in the long-term behavior of interacting economic variables, especially when exogenous variables can be applied to act on the system. It also illustrates the dynamic behavior of economic variables, showing that economic systems, and financial markets in particular, behave as complex adaptive systems and can be modeled as such. In these regards, the objectives of the research were met.

6.3 Contributions

This research makes several unique contributions to the fields of economics and operations research. The field of complex systems science was extended further into the realm of economic studies, beyond that of previous systems dynamics models that were not examined here. In doing so, the logistic differential equation, expanded and applied to a system of differential equations, was used to replicate a financial market as a complex adaptive system, laying the foundation for future studies in this area. The same system of differential equations was used to solve an inverse problem in an economic context, and insight was obtained regarding the effect of Federal Reserve interest rates on stock market performance. Finally, the strength and importance of relationships between the model components were quantified. While the exact values of the coefficients are unique to the model and its data, the proportional quantities of these coefficients are indicators of the components' influence on one another.

From a national policy perspective, the model increases understanding of how interest rates influence the economy over extended periods of time. These are insights valuable both to the Federal Reserve that sets interests rates, as well as to private and public economists interested in the effects of interest rates over time. Expanding the pilot model to include other variables would provide even more information, potentially showing the impact of multiple forcing functions on different aspects of the economy. Additionally, as briefly shown, the model can be used to identify turning points in the economy that could further inform policy making decisions.

Perhaps most importantly, the pilot model demonstrated the ability to make long-range forecasts. As was shown with the Euler curves and discussed earlier, the model is capable of making predictions many years into the future based only on a given starting point and its own past predictions. Unlike ARIMA forecasting models whose prediction intervals grow increasingly wider as the forecasts project into the future, the model developed here was shown to maintain relatively narrow intervals, even after having predicted 17 years into the future. This is certainly an advantage over traditional forecasting methods, and its demonstrated use here is an important contribution that could help inform policy makers when faced with decisions that have long-term ramifications.

With increased understanding also comes increased utility. Advances in the knowledge of exogenous influencing factors and their impact on economic variables shows how these factors can be used either coercively or defensively. As the United States faces rising near-peer adversaries, additional economic tools provides leaders with new options without resorting to expensive and politically or diplomatically dangerous conflicts.

6.4 Recommendations for Future Research

Because they are only abstractions of reality, all models are incorrect representations of the real world. This model is no different, and future work is still needed before it can serve as an accurate tool for describing all economic phenomena and precisely predicting future outcomes. Nonetheless, it constitutes a foundation for future research, several ideas of which are listed below.

1. **Comparison to autoregression.** Traditional time series analysis, such as moving averages, smoothing methods, and autoregressive integrated moving averages (ARIMA), relies on autoregression or historical trends to forecast future performance. These methods work well in steady systems with readily identifiable factors, and models can be built with respectably high R-squared values that indicate a good fit. For this reason they are frequently used by government and private economists, and they constitute the foundation of most econometric modeling efforts. However, because of the erratic and

dynamic nature of the markets, they may not predict financial markets well. Forecasts are only accurate over short time periods and rely on direct correlations, or autocorrelations, between variables, resulting in mostly linear cause-and-effect relationships. They do not address feedback between the model components. For this reason, economic behavior may be better modeled as a complex system. A direct comparison of these two methods was beyond the scope of this research, but it would provide valuable evidence in establishing which method is best for making accurate predictions on future behavior and testing hypothetical scenarios.

2. **Model refinement.** The variables included in the model were chosen using a "model-driven" approach rather than being "data-driven." In other words, while extensive investigation was performed to identify factors that contribute to stock market performance, the model was ultimately conjectured based on the results of the literature review and basic portfolio theory. It was further developed by testing different variables to see which ones produced the most significant model coefficients and fit the model to the actual data best. Less emphasis was placed on its predictive, or forecasting, ability, although this was certainly considered during in Section 5.3 when the model was fitted using the three different approaches.

   The number of model components was limited in order to maintain simplicity, hoping that parsimony would also be achieved. Consequently, only those variables considered most important to the model were retained. Innumerable other models could be contrived however, possibly generating better results than were obtained here. Additionally, because the stock market is dynamic, certain variables could be more significant in one time period but less significant in another. In these situations, research and experimentation are required to determine which variables are most appropriate for the time period of interest.

Additionally, future models should examine whether a fully-connected network model is appropriate. It is possible that not all variables in the model influence the others, and less significant relationships might be dropped. Doing so may result in better model fits and increased simplicity.

3. **Extending the model.** This research focused on stock market dynamics, motivated in part by observed market responses to interest rate fluctuations. The U.S. economy is extremely multifaceted however, and a derivation of this model could be applied to a variety of different circumstances to which it may be better suited. Suggestions include:

    a. The relationships between asset classes. Investors generally seek to maximize their risk-adjusted returns. Often, decreases in one asset class, such as bonds, are offset by gains in another class, like stocks. Other major asset classes include real estate, money market funds, and commodities. Within these categories are further stratifications, within which investors will choose in order to diversify their portfolios. For example, stocks are classified as small, mid, and large cap; emerging market or developed market. The flow of capital between these classes may parallel movements of the greater national and global economies, and are subject to similar influence factors such as interest rates, exchange rates, and the regulatory environment. This flow of capital could be modeled using an approach similar to the method used in this thesis, but with different variables and influencing factors.

b. The relationships between national accounts. From an expenditure standpoint, U.S. gross domestic product is calculated by summing the aggregate values of national consumption, income, government expenditures, and net exports. Because of the complementary nature of these accounts, increases in one area may lead to increases in another area in a cyclic pattern. Similarly, decreases in one may cause proportional decreases in the others. National production is closely tied to labor participation rates, unemployment, and inflation, which the Federal Reserve is charged with controlling via monetary policy. Abundant data is available on the national accounts, and the levels of each could be modeled using a system of differential equations, resulting in a model of the overall economy.

c. International trade, deficits, and sanctions. Trade between nations is very dynamic and influenced by a multitude of factors. Normally, a nation prefers to maintain a positive trade deficit, meaning they are exporting more than importing and gaining a net profit from trading activities. Consumer economies such as the United States frequently have negative trade deficits however, purchasing foreign-manufactured goods on credit. Nations utilize many techniques to influence trade. Free trade agreements simply allow countries to trade without tariffs or restrictions. Manufacturing nations will sometimes devalue their currency to maintain favorable trading conditions with other nations: an American dollar will buy more in China than the yuan will in the United States, thus leading to greater imports into the United States from China. Also, sanctions will be placed on competing countries in order to restrict their trade, either as a punitive action or as a protection against foreign encroachment on domestic business. Trade levels between countries could be modeled as a complex adaptive system, where domestic and foreign interest rates mutually affect exchange rates and purchasing power. Sanctions, tariffs, and other trade restrictions could be included in the model as additional influencing factors, and illegal activities such as smuggling could also be included as a model component, using the best available data on such activities.

Because the model is only a proof of concept, ample opportunity exists to expand upon it, using the suggestions listed here. Comparing the model to more conventional forecasting models would establish its relative accuracy and usefulness in particular applications. After determining its comparative utility, it should be refined in order to enhance its predictive ability, and variations of the model could be created to address specific areas of interest, such as capital fund flows, domestic economic growth and international trade.

6.5 Conclusion

This research has analyzed a very old problem in a new light. Whereas conventional methods of modeling the economy may have been well tailored to historical circumstances where markets were segmented and communication slower, today's world is different and requires new tools. Modern technology is making world economies more globalized, and financial markets are becoming more interdependent and connected. Trading partners that were once hindered by long distances can now communicate and execute transactions at the speed of light, while investors and speculators can act on perceived opportunities instantaneously. This makes capital transfers and fund flows faster and more liquid than ever before. It also makes economic behavior more dynamic and fluid.

Representing the economy and its components as a complex adaptive system is more germane to this new economic environment than traditional methods. A model using a system of differential equations may better reflects the nature of real systems, whose components grow and change in complementary ways in reaction to the pressures and stimuli to which they are exposed. The pilot model presented in this research serves as one example of how these fluctuations can be accurately simulated and experimented upon.

While significant research remains to be done, the proposed model constitutes a potential new tool for understanding economic systems and assessing the impact of exogenous influences on system components. As such, it may provide a path toward a more stable and prosperous future that anticipates change rather than reacts to it.

Appendix A. Data Tables

Appendix A contains all the data used in this research. **Table A.1** contains the data used for the seven model components after adjustment for inflation using the Consumer Price Index and the procedure outlined in Section 4.3.1. Table A.2 provides the scalars used to scale each of the seven data series so the maximum value of each variable is equal to 80, as discussed in Section 4.3.2. **Table A.3** shows the data for the seven model components after applying the scalars in Table A.2. **Table A.4** and **Table A.5** are the unscaled and scaled versions of the hypothetical interest rates used in the “what if” scenarios of Section 5.7. **Table A.6** is the Consumer Price Index data used to adjust the six endogenous model variables to April 2015 dollars.

Table A.1: Unscaled Model Data

Quarter	GDP Per Capita[1] (Seasonally Adjusted, 2015 Dollars)	As Reported Earnings[2] (Bil, 2015 Dollars)	Dividends[3] (Bil, 2015 Dollars)	Buybacks[4] (Bil, 2015 Dollars)	S&P 500 Estimated Retained Quarterly Earnings[5] (Bil, 2015 Dollars)	S&P 500[6] (End of Qtr, 2015 Dollars)	Federal Funds Rate[7] (Qtrly Avg)
Jan-98	45476.77	113.323	41.4086	37.10397	34.81044	1405.422	5.52
Apr-98	45632.2	109.0392	46.17871	41.08754	21.77295	1505.471	5.5
Jul-98	45982.63	99.8616	47.3204	53.34789	-0.80669	1627.123	5.53
Oct-98	46483.3	95.73165	44.73442	44.80413	6.193102	1583.321	4.86
Jan-99	46765.15	123.311	45.11653	46.65711	31.53738	1633.638	4.73
Apr-99	46956.55	140.2446	46.86031	43.54035	49.84396	1825.491	4.75
Jul-99	47318.83	133.7828	49.90222	41.66169	42.2189	1909.606	5.1
Oct-99	47945.39	145.0666	46.0078	61.31008	37.74868	1915.652	5.3

[1] (U.S. Bureau of Economic Analysis, 2015)
[2] (S&P Dow Jones Indices, 2015; Personal communication with H. Silverblatt, 2015)
[3] (S&P Dow Jones Indices, 2015; Personal communication with H. Silverblatt, 2015)
[4] (S&P Dow Jones Indices, 2015; Personal communication with H. Silverblatt, 2015)
[5] Calculated by subtracting Dividends and Buybacks from As Reported Earnings
[6] (S&P Dow Jones Indices, 2015; Personal communication with H. Silverblatt, 2015)
[7] (Board of Governors of the Federal Reserve System (US), 2015)

Table A.1: Unscaled Model Data

(Continued)

Quarter	GDP Per Capita[1] (Seasonally Adjusted, 2015 Dollars)	As Reported Earnings[2] (Bil, 2015 Dollars)	Dividends[3] (Bil, 2015 Dollars)	Buybacks[4] (Bil, 2015 Dollars)	S&P 500 Estimated Retained Quarterly Earnings[5] (Bil, 2015 Dollars)	S&P 500[6] (End of Qtr, 2015 Dollars)	Federal Funds Rate[7] (Qtrly Avg)
Jan-00	48014.85	156.6187	46.50687	65.39581	44.71605	1947.741	5.68
Apr-00	48769.52	154.8065	47.31474	49.97387	57.51787	1992.782	6.27
Jul-00	48688.38	159.8296	47.68074	41.16569	70.98314	2011.323	6.52
Oct-00	48802.66	106.3283	46.65784	44.5149	15.15556	2034.417	6.47
Jan-01	48503.52	107.7665	44.37446	40.94209	22.45	1873.811	5.6
Apr-01	48690.62	56.71084	45.08688	43.83254	-32.2086	1733.147	4.33
Jul-01	48238.79	61.39163	48.59682	44.83578	-32.041	1657.461	3.5
Oct-01	48079.04	63.89227	46.65894	42.12946	-24.8961	1534.142	2.13
Jan-02	48304.95	107.6076	44.14371	38.97923	24.48467	1504.07	1.73
Apr-02	48396.35	80.3062	48.51102	39.4341	-7.63892	1518.05	1.75
Jul-02	48470.84	99.59574	45.54783	44.54478	9.503133	1417.942	1.74
Oct-02	48425.97	34.84328	49.43101	38.58783	-53.1756	1182.333	1.44
Jan-03	48713.67	138.1794	45.46474	37.72046	54.99422	1169.735	1.25
Apr-03	49128.66	128.5756	47.32971	35.56608	45.67978	1121.572	1.25
Jul-03	49904.55	145.0426	49.91037	42.636	52.49626	1218.892	1.02
Oct-03	50477.41	151.6711	58.25968	48.00243	45.40898	1293.737	1
Jan-04	50865.79	174.8386	52.52068	53.21444	69.10347	1365.866	1
Apr-04	51247.27	174.9728	53.51299	52.31439	69.14542	1452.153	1.01
Jul-04	51683.11	162.3026	55.8788	56.04611	50.37771	1418.596	1.43
Oct-04	52048.34	158.3476	60.59018	80.9963	16.76116	1389.88	1.95
Jan-05	52651.39	187.1589	59.36879	99.44563	28.34445	1453.803	2.47
Apr-05	52947.94	201.685	59.14916	98.22219	44.31362	1482.219	2.94
Jul-05	53577.88	188.5718	58.90294	97.9536	31.71529	1449.98	3.46
Oct-05	53799.59	186.2988	65.48475	124.5537	-3.73974	1484.122	3.98
Jan-06	54413.35	210.4563	63.35456	118.9152	28.1865	1484.398	4.45
Apr-06	54446.14	214.2993	64.19356	137.3432	12.76246	1539.288	4.91
Jul-06	54362.81	225.9052	64.40054	128.4085	33.09621	1512.275	5.25
Oct-06	54557.47	211.3246	71.8846	166.2792	-26.8392	1511.552	5.24
Jan-07	54762.38	220.6169	67.6939	136.1325	16.7905	1644.133	5.25
Apr-07	55097.45	223.6413	68.77922	181.5768	-26.7147	1669.628	5.25
Jul-07	55232.82	153.0552	70.09426	196.8967	-113.936	1719.653	5.07
Oct-07	55142.47	77.92553	76.28425	161.1383	-159.497	1708.653	4.5
Jan-08	54596.74	152.7642	69.7171	128.6672	-45.6201	1700.448	3.18
Apr-08	54775.57	126.1374	69.66985	98.8807	-42.4131	1519.477	2.08
Jul-08	54402.83	94.91865	68.6575	100.2555	-73.9943	1510.483	1.94
Oct-08	53074.81	-225.312	69.32828	53.64158	-348.282	1362.592	0.51
Jan-09	52143.67	72.48539	57.42925	34.17414	-19.118	1018.674	0.18

Table A.1: Unscaled Model Data

(Continued)

Quarter	GDP Per Capita[1] (Seasonally Adjusted, 2015 Dollars)	As Reported Earnings[2] (Bil, 2015 Dollars)	Dividends[3] (Bil, 2015 Dollars)	Buybacks[4] (Bil, 2015 Dollars)	S&P 500 Estimated Retained Quarterly Earnings[5] (Bil, 2015 Dollars)	S&P 500[6] (End of Qtr, 2015 Dollars)	Federal Funds Rate[7] (Qtrly Avg)
Apr-09	51616.39	130.5804	52.60912	26.72406	51.24724	911.2077	0.18
Jul-09	51501.42	143.5633	51.99104	38.3762	53.1961	993.7135	0.15
Oct-09	51768.54	148.0688	53.72913	79.22622	15.1135	1102.669	0.12
Jan-10	52064.24	172.9139	54.00098	60.53464	58.3783	1202.11	0.13
Apr-10	52636.84	194.7508	55.18928	84.9457	54.61579	1233.428	0.19
Jul-10	52997.6	192.9686	55.9431	86.83147	50.19403	1242.576	0.19
Oct-10	53353.67	204.5013	59.70806	94.0167	50.77658	1198.912	0.19
Jan-11	53048.92	211.4803	60.771	97.35591	53.3534	1312.704	0.15
Apr-11	53451.94	218.2454	63.63856	117.7729	36.83392	1402.661	0.09
Jul-11	53434.91	220.751	63.41593	126.8426	30.49247	1395.874	0.08
Oct-11	53720.85	199.0006	70.20485	97.6158	31.17991	1294.494	0.07
Jan-12	53987.07	220.635	67.91496	89.34842	63.37162	1293.751	0.1
Apr-12	54104.13	205.875	70.96116	117.8135	17.10027	1410.901	0.15
Jul-12	54134.85	199.1391	72.96132	108.9184	17.25934	1402.511	0.14
Oct-12	54005.11	192.849	83.44284	103.6344	5.771745	1453.055	0.16
Jan-13	54103.02	224.3346	73.67289	103.9395	46.72217	1469.203	0.14
Apr-13	54148.46	229.752	79.50694	122.4155	27.82957	1559.684	0.12
Jul-13	54443.1	226.1801	81.80842	132.281	12.09068	1648.937	0.09
Oct-13	54844.85	242.8612	87.34084	133.0001	22.52026	1709.634	0.09
Jan-14	54601.45	226.9775	83.86973	162.9797	-19.872	1811.82	0.07
Apr-14	55124.23	246.0397	88.18211	118.2261	39.63144	1862.902	0.09
Jul-14	55622.96	247.4501	90.28284	147.2441	9.92315	1907.459	0.09
Oct-14	55607.27	211.8532	93.76346	132.8925	-14.8027	1981.112	0.1
Jan-15	55386.71	193.7925	93.98457	145.0223	-45.2144	2033.887	0.11

Table A.2: Data Scalars for Table A.3

Economic Variable	Scalar
GDP Per Capita	0.0014382550803589
As Reported Earnings	0.3232975178375920
Dividends	0.8512035407247510
Buybacks	0.4063043445781700
Retained Quarterly Earnings	1.1270282311652800
S&P 500	0.0393232953811036
Federal Funds Rate	4.4969083754918500

Table A.3: Scaled Model Data

Quarter	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500	Federal Funds Rate
Jan-98	65.40719985	36.63705	35.24715	15.0755	39.23235	55.26584	24.822934
Apr-98	65.63073813	35.2521	39.30748	16.69404	24.53873	59.20009	24.732996
Jul-98	66.13475374	32.28501	40.27929	21.67548	-0.90916	63.98385	24.867903
Oct-98	66.85484357	30.94981	38.0781	18.20411	6.979801	62.26141	21.854975
Jan-99	67.26022112	39.86615	38.40335	18.95699	35.54351	64.24002	21.270377
Apr-99	67.53550285	45.34074	39.88767	17.69063	56.17555	71.78432	21.360315
Jul-99	68.0565527	43.25165	42.47695	16.92733	47.58189	75.09199	22.934233
Oct-99	68.95769539	46.89966	39.162	24.91055	42.54383	75.32975	23.833614
Jan-00	69.05760245	50.63445	39.58682	26.5706	50.39625	76.59159	25.54244
Apr-00	70.14301239	50.04855	40.27447	20.3046	64.82426	78.36276	28.195616
Jul-00	70.02630753	51.6725	40.58601	16.7258	80	79.09185	29.319843
Oct-00	70.19067084	34.37567	39.71532	18.0866	17.08074	80	29.094997
Jan-01	69.76043666	34.84065	37.77169	16.63495	25.30178	73.68442	25.182687
Apr-01	70.02952726	18.33447	38.37811	17.80935	-36.3	68.15304	19.471613
Jul-01	69.37969027	19.84776	41.36578	18.21697	-36.1111	65.17682	15.739179
Oct-01	69.14992191	20.65621	39.71625	17.11738	-28.0586	60.32751	9.5784148
Jan-02	69.47483809	34.78927	37.57528	15.83743	27.59492	59.145	7.7796515
Apr-02	69.60629223	25.96279	41.29275	16.02224	-8.60928	59.69472	7.8695897
Jul-02	69.71343805	32.19906	38.77047	18.09874	10.7103	55.75814	7.8246206
Oct-02	69.64889156	11.26474	42.07585	15.6784	-59.9304	46.49323	6.4755481
Jan-03	70.06268766	44.67306	38.69975	15.32599	61.98004	45.99782	5.6211355
Apr-03	70.65954541	41.56816	40.28721	14.45065	51.4824	44.10392	5.6211355
Jul-03	71.77547401	46.89192	42.48388	17.32319	59.16477	47.93087	4.5868465
Oct-03	72.59939715	49.03489	49.59084	19.5036	51.17721	50.87402	4.4969084
Jan-04	73.15797388	56.52488	44.70579	21.62126	77.88156	53.71035	4.4969084
Apr-04	73.70664182	56.56827	45.55045	21.25557	77.92884	57.10344	4.5418775
Jul-04	74.33349662	52.47203	47.56423	22.77178	56.7771	55.78387	6.430579
Oct-04	74.8587962	51.1934	51.57458	32.90915	18.8903	54.65465	8.7689713
Jan-05	75.72613158	60.508	50.53493	40.40519	31.94499	57.16834	11.107364
Apr-05	76.15264762	65.20425	50.34798	39.9081	49.9427	58.28573	13.220911
Jul-05	77.05866258	60.96481	50.13839	39.79897	35.74402	57.01799	15.559303
Oct-05	77.37752965	60.22992	55.74085	50.60673	-4.21479	58.36058	17.897695
Jan-06	78.26027891	68.04	53.92763	48.31577	31.76698	58.37142	20.011242
Apr-06	78.30743313	69.28242	54.64179	55.80315	14.38365	60.52989	22.07982
Jul-06	78.18759259	73.03459	54.81797	52.17291	37.30036	59.46763	23.608769

Table A.3: Scaled Model Data

(Continued)

Quarter	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500	Federal Funds Rate
Oct-06	78.46756179	68.32073	61.18843	67.55996	-30.2485	59.43921	23.5638
Jan-07	78.7622752	71.32488	57.62129	55.31121	18.92337	64.65274	23.608769
Apr-07	79.24419403	72.30266	58.54511	73.77544	-30.1083	65.65528	23.608769
Jul-07	79.43888021	49.48237	59.66448	80	-128.409	67.62241	22.799325
Oct-07	79.30893539	25.19313	64.93343	65.47118	-179.758	67.18987	20.236088
Jan-08	78.52403373	49.38829	59.34344	52.27804	-51.4151	66.86721	14.300169
Apr-08	78.78124235	40.77992	59.30322	40.17566	-47.8008	59.75084	9.3535694
Jul-08	78.24514568	30.68696	58.44151	40.73423	-83.3937	59.39718	8.7240022
Oct-08	76.33512153	-72.8428	59.01248	21.79481	-392.524	53.58159	2.2934233
Jan-09	74.99590155	23.43435	48.88398	13.8851	-21.5465	40.05761	0.8094435
Apr-09	74.23754171	42.21633	44.78107	10.8581	57.75708	35.83169	0.8094435
Jul-09	74.07218455	46.41367	44.25496	15.59242	59.9535	39.07609	0.6745363
Oct-09	74.45635938	47.87029	45.73443	32.18996	17.03334	43.36057	0.539629
Jan-10	74.88165738	55.90264	45.96583	24.59549	65.794	47.27092	0.5845981
Apr-10	75.70519856	62.96244	46.97731	34.51381	61.55354	48.50246	0.8544126
Jul-10	76.22406112	62.38627	47.61896	35.28	56.57008	48.8622	0.8544126
Oct-10	76.73618094	66.11477	50.82371	38.19939	57.22664	47.14518	0.8544126
Jan-11	76.29788184	68.37106	51.72849	39.55613	60.13079	51.61986	0.6745363
Apr-11	76.87752746	70.55821	54.16937	47.85166	41.51287	55.15726	0.4047218
Jul-11	76.85303212	71.36826	53.97986	51.53672	34.36587	54.89037	0.3597527
Oct-11	77.26429215	64.33638	59.75862	39.66172	35.14063	50.90378	0.3147836
Jan-12	77.64718224	71.33075	57.80945	36.30265	71.4216	50.87456	0.4496908
Apr-12	77.81554303	66.55887	60.40239	47.86815	19.27248	55.48126	0.6745363
Jul-12	77.85972352	64.38116	62.10494	44.25402	19.45177	55.15137	0.6295672
Oct-12	77.67312935	62.3476	71.02684	42.10711	6.50492	57.13891	0.7195053
Jan-13	77.81394189	72.52682	62.71063	42.23109	52.6572	57.77389	0.6295672
Apr-13	77.87929518	74.27825	67.67659	49.73795	31.36471	61.33193	0.539629
Jul-13	78.30307086	73.12347	69.63561	53.74635	13.62654	64.84162	0.4047218
Oct-13	78.88087992	78.51643	74.34483	54.03852	25.38097	67.22845	0.4047218
Jan-14	78.53081944	73.38125	71.39021	66.21936	-22.3963	71.24673	0.3147836
Apr-14	79.28270132	79.54402	75.06093	48.03579	44.66575	73.25546	0.4047218
Jul-14	80	80	76.84907	59.82592	11.18367	75.00759	0.4047218
Oct-14	79.97744058	68.49163	79.81179	53.99478	-16.683	77.90385	0.4496908
Jan-15	79.6602232	62.65263	80	58.9232	-50.9579	79.97915	0.4946599

Table A.4: Unscaled Hypothetical Federal Funds Rates

Quarter	Scenario 1	Scenario 2	Scenario 3
Jan-98	5.52	5.52	5.52
Apr-98	5.5	5.5	5.5
Jul-98	5.53	5.53	5.53
Oct-98	4.86	4.86	4.86
Jan-99	4.73	4.73	4.73
Apr-99	4.75	4.75	4.75
Jul-99	5.1	5.1	5.1
Oct-99	5.3	5.3	5.3
Jan-00	5.68	5.68	5.68
Apr-00	6.27	6.27	6.27
Jul-00	6.52	6.52	6.52
Oct-00	6.47	6.47	6.47
Jan-01	5.6	5.6	5.6
Apr-01	4.33	4.33	4.33
Jul-01	3.5	3.5	3.5
Oct-01	2.13	2.13	2.13
Jan-02	1.73	1.73	1.73
Apr-02	1.75	1.75	1.75
Jul-02	1.74	1.74	1.74
Oct-02	1.44	1.44	1.44
Jan-03	1.25	1.25	1.25
Apr-03	1.25	1.25	1.25
Jul-03	1.02	1.02	1.02
Oct-03	1	1	1
Jan-04	1	1	1
Apr-04	1.01	1.01	1.01
Jul-04	1.43	1.43	1.43
Oct-04	1.95	1.95	1.95
Jan-05	2.47	2.47	2.47
Apr-05	2.94	2.94	2.94
Jul-05	3.46	3.46	3.46
Oct-05	3.98	3.98	3.98
Jan-06	4.45	4.45	4.45
Apr-06	4.91	4.91	4.91
Jul-06	5.25	5.25	5.25
Oct-06	5.24	5.24	5.24
Jan-07	5.25	5.25	5.25

Table A.4: Unscaled Hypothetical Federal Funds Rates

(Continued)

Quarter	Scenario 1	Scenario 2	Scenario 3
Apr-07	5.25	5.25	5.25
Jul-07	5.07	5.07	5.07
Oct-07	4.5	4.5	4.75
Jan-08	3.18	3.18	4.5
Apr-08	2.08	2.08	4.25
Jul-08	1.94	2	4
Oct-08	0.51	2	3.75
Jan-09	0.18	2	3.5
Apr-09	0.18	2	3.25
Jul-09	0.3	2	3
Oct-09	0.45	2	2.75
Jan-10	0.6	2	2.5
Apr-10	0.75	2	2.25
Jul-10	0.9	2	2
Oct-10	1.05	2	2
Jan-11	1.2	2	2
Apr-11	1.35	2	2
Jul-11	1.5	2	2
Oct-11	1.65	2	2
Jan-12	1.8	2	2
Apr-12	1.95	2	2
Jul-12	2.1	2	2
Oct-12	2.25	2	2.25
Jan-13	2.4	2	2.5
Apr-13	2.55	2	2.75
Jul-13	2.7	2	3
Oct-13	2.85	2	3.25
Jan-14	3	2	3.5
Apr-14	3.15	2	3.75
Jul-14	3.3	2	4
Oct-14	3.45	2	4.25
Jan-15	3.6	2	4.5

Table A.5: Scaled Hypothetical Federal Funds Rates

Quarter	Scenario 1	Scenario 2	Scenario 3
Jan-98	24.822934	24.822934	24.822934
Apr-98	24.732996	24.732996	24.732996
Jul-98	24.867903	24.867903	24.867903
Oct-98	21.854975	21.854975	21.854975
Jan-99	21.270377	21.270377	21.270377
Apr-99	21.360315	21.360315	21.360315
Jul-99	22.934233	22.934233	22.934233
Oct-99	23.833614	23.833614	23.833614
Jan-00	25.54244	25.54244	25.54244
Apr-00	28.195616	28.195616	28.195616
Jul-00	29.319843	29.319843	29.319843
Oct-00	29.094997	29.094997	29.094997
Jan-01	25.182687	25.182687	25.182687
Apr-01	19.471613	19.471613	19.471613
Jul-01	15.739179	15.739179	15.739179
Oct-01	9.5784148	9.5784148	9.5784148
Jan-02	7.7796515	7.7796515	7.7796515
Apr-02	7.8695897	7.8695897	7.8695897
Jul-02	7.8246206	7.8246206	7.8246206
Oct-02	6.4755481	6.4755481	6.4755481
Jan-03	5.6211355	5.6211355	5.6211355
Apr-03	5.6211355	5.6211355	5.6211355
Jul-03	4.5868465	4.5868465	4.5868465
Oct-03	4.4969084	4.4969084	4.4969084
Jan-04	4.4969084	4.4969084	4.4969084
Apr-04	4.5418775	4.5418775	4.5418775
Jul-04	6.430579	6.430579	6.430579
Oct-04	8.7689713	8.7689713	8.7689713
Jan-05	11.107364	11.107364	11.107364
Apr-05	13.220911	13.220911	13.220911
Jul-05	15.559303	15.559303	15.559303
Oct-05	17.897695	17.897695	17.897695
Jan-06	20.011242	20.011242	20.011242
Apr-06	22.07982	22.07982	22.07982
Jul-06	23.608769	23.608769	23.608769
Oct-06	23.5638	23.5638	23.5638
Jan-07	23.608769	23.608769	23.608769

Table A.5: Scaled Hypothetical Federal Funds Rates

(Continued)

Quarter	Scenario 1	Scenario 2	Scenario 3
Apr-07	23.608769	23.608769	23.608769
Jul-07	22.799325	22.799325	22.799325
Oct-07	20.236088	20.236088	21.360315
Jan-08	14.300169	14.300169	20.236088
Apr-08	9.3535694	9.3535694	19.111861
Jul-08	8.7240022	8.9938168	17.987634
Oct-08	2.2934233	8.9938168	16.863406
Jan-09	0.8094435	8.9938168	15.739179
Apr-09	0.8094435	8.9938168	14.614952
Jul-09	1.3490725	8.9938168	13.490725
Oct-09	2.0236088	8.9938168	12.366498
Jan-10	2.698145	8.9938168	11.242271
Apr-10	3.3726813	8.9938168	10.118044
Jul-10	4.0472175	8.9938168	8.9938168
Oct-10	4.7217538	8.9938168	8.9938168
Jan-11	5.3962901	8.9938168	8.9938168
Apr-11	6.0708263	8.9938168	8.9938168
Jul-11	6.7453626	8.9938168	8.9938168
Oct-11	7.4198988	8.9938168	8.9938168
Jan-12	8.0944351	8.9938168	8.9938168
Apr-12	8.7689713	8.9938168	8.9938168
Jul-12	9.4435076	8.9938168	8.9938168
Oct-12	10.118044	8.9938168	10.118044
Jan-13	10.79258	8.9938168	11.242271
Apr-13	11.467116	8.9938168	12.366498
Jul-13	12.141653	8.9938168	13.490725
Oct-13	12.816189	8.9938168	14.614952
Jan-14	13.490725	8.9938168	15.739179
Apr-14	14.165261	8.9938168	16.863406
Jul-14	14.839798	8.9938168	17.987634
Oct-14	15.514334	8.9938168	19.111861
Jan-15	16.18887	8.9938168	20.236088

Table A.6: Consumer Price Index for All Urban Consumers: All Items Less Food & Energy

Quarter	Consumer Price Index[8]
Jan-98	171.900
Apr-98	172.867
Jul-98	173.900
Oct-98	174.867
Jan-99	175.633
Apr-99	176.467
Jul-99	177.400
Oct-99	178.400
Jan-00	179.567
Apr-00	180.700
Jul-00	181.900
Oct-00	183.000
Jan-01	184.333
Apr-01	185.467
Jul-01	186.733
Oct-01	187.967
Jan-02	189.000
Apr-02	189.967
Jul-02	190.967
Oct-02	191.833
Jan-03	192.467
Apr-03	192.800
Jul-03	193.567
Oct-03	194.067
Jan-04	195.000
Apr-04	196.233
Jul-04	197.067
Oct-04	198.267
Jan-05	199.500
Apr-05	200.433
Jul-05	201.100
Oct-05	202.433
Jan-06	203.700
Apr-06	205.367
Jul-06	206.767
Oct-06	207.833
Jan-07	209.051
Apr-07	210.066
Jul-07	211.149

[8] (U.S. Bureau of Labor Statistics, 2015)

Table A.6: Consumer Price Index for All Urban Consumers: All Items Less Food & Energy

(Continued)

Quarter	Consumer Price Index[8]
Oct-07	212.635
Jan-08	214.043
Apr-08	214.973
Jul-08	216.357
Oct-08	216.887
Jan-09	217.797
Apr-09	218.907
Jul-09	219.560
Oct-09	220.683
Jan-10	220.716
Apr-10	220.993
Jul-10	221.528
Oct-10	222.107
Jan-11	223.114
Apr-11	224.277
Jul-11	225.715
Oct-11	226.917
Jan-12	228.109
Apr-12	229.336
Jul-12	230.251
Oct-12	231.319
Jan-13	232.545
Apr-13	233.167
Jul-13	234.250
Oct-13	235.263
Jan-14	236.294
Apr-14	237.584
Jul-14	238.414
Oct-14	239.290
Jan-15	240.304
Apr-15	241.787

Appendix B. Model Coefficients

Appendix B includes all the coefficients for the A, B, and D matrices, as found in Chapter 5, for the weighted and unweighted models fit to the data, slopes and secants. As mentioned in Section 5.3.1, these coefficients are those used to analyze the model and its results, out to 15 digits.

Table B.1: Section 5.3.1 weighted data model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.0046317138188040	0.0174647402036114	0.0012837068483229	-0.0135377666732315	-0.0005596410136027	-0.0031274091040255
As Reported Earnings	0.0661448647383908	0.0235381145233712	-0.0898822435684387	-0.0749548852539955	-0.0043624932677409	-0.1716069878739070
Dividends	0.0148849064293154	0.0357280974620716	-0.0610709823401240	0.0056209881996505	0.0002991487309807	0.0038787681340801
Buybacks	0.0582522296311235	0.0366181879749865	-0.0165583784338951	-0.0088239934905792	0.0155523545372852	0.0673211844084290
Retained Quarterly Earnings	0.2232413093014860	-0.1034878685888760	-0.0215943805571861	-0.4342138153787130	-0.0798842595078615	-0.6632716335804210
S&P 500	-0.0098187038139623	0.0600808735759763	-0.0217467823839637	-0.0325262001035667	-0.0047185721648729	-0.0992121307931393

Table B.2: Section 5.3.1 weighted data model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.5002405473870150	0.5000931046405600	0.5001256660856510	0.5008196624945550	0.5004385779568260	0.5026050358149460
As Reported Earnings	0.4676895249600370	0.4900488327226660	0.4932582672725760	0.5707092398639840	0.4874503504364030	0.5221778067688980
Dividends	0.4954944850714840	0.4923702070689320	0.5035486451344100	0.4950935093635560	0.5004609584666960	0.5387180355922510
Buybacks	0.6243525592253500	0.6045141794343480	0.4843303644924010	0.5638881515332220	0.6167830280647030	1.0000000000000000
Retained Quarterly Earnings	0.3178152002541060	0.6625102166338840	0.5434334008251030	0.5199262271696710	0.5062598086660270	0.3233360047717670
S&P 500	0.4934140452865000	0.4574545819239820	0.5415921245598930	0.4246224960359800	0.4958604088783630	0.4854962200735490

Table B.3: Section 5.3.1 weighted data model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0131907213674146
As Reported Earnings	0.0000000000000000
Dividends	-0.0142205310197924
Buybacks	0.0633472425816279
Retained Quarterly Earnings	1.3350914109501200
S&P 500	0.2310488113876760

Table B.4: Section 5.3.2 unweighted data model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.0048268027010570	0.0034999878473108	0.0007801018681323	-0.0018352116103390	0.0034776281936657	0.0009137958501261
As Reported Earnings	-0.0268321693070791	-0.0114290138977363	0.0139613721838668	-0.0793760036429644	0.0087039047651542	-0.1964775049322260
Dividends	0.0092660475665472	0.0103996828394200	0.0002337036041175	0.0017108361302597	0.0079650828658315	0.0023414076545176
Buybacks	0.0335972994202317	0.0195954175897718	0.0081606043100029	-0.0022925360710268	0.0235718970284039	-0.0108859109456590
Retained Quarterly Earnings	-0.1431040330038740	0.0268762665849557	0.0639189798940280	-0.2332114262154060	-0.0630173014781095	-0.1335606657566990
S&P 500	-0.0052830213544803	0.0215691696928831	-0.0010013616285504	0.0099471311412554	0.0125925841753081	-0.0098206023698460

Table B.5: Section 5.3.2 unweighted data model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.4994799013934560	0.4960812522160910	0.4985546670467230	0.5010647122989820	0.4898392930218560	0.4946020554851310
As Reported Earnings	0.5325082479133940	0.5567964891805910	0.4830947463009010	0.8416794139355110	0.5022300136997260	0.6880482677496820
Dividends	0.4966234763788250	0.4602327383652080	0.5024628144744490	0.4861738239013330	0.4726929495262910	0.4934911912682340
Buybacks	0.2928258545095310	0.6353800492279700	0.5647754844451760	0.5065502504603330	0.4256122929484710	0.1455324774391580
Retained Quarterly Earnings	0.2274312364053160	0.9999999984938380	0.0429731530202502	0.2077254392275090	0.4508044042851970	0.2748454934275290
S&P 500	0.2669827575397010	0.3894465369323180	0.5184461043298300	0.3902547152179480	0.5023531457316450	0.7686320653078560

Table B.6: Section 5.3.2 unweighted data model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0013251574863015
As Reported Earnings	0.0000000000000000
Dividends	-0.0001774532338473
Buybacks	-0.0017527851803371
Retained Quarterly Earnings	0.0032350741264206
S&P 500	0.0686249909077331

Table B.7: Section 5.4 weighted slope model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.0072100000000000	0.0187400000000000	0.0016000000000000	-0.0168300000000000	-0.0020700000000000	-0.0054400000000000
As Reported Earnings	0.0291200000000000	-0.0158400000000000	-0.0534200000000000	-0.2082300000000000	-0.0169200000000000	-0.5661600000000000
Dividends	0.0153000000000000	0.0412800000000000	-0.0816300000000000	0.0027200000000000	-0.0015200000000000	0.0037400000000000
Buybacks	0.0596500000000000	0.0151600000000000	-0.0205800000000000	-0.0202600000000000	0.0234700000000000	-0.0529100000000000
Retained Quarterly Earnings	-5.3566600000000000	-0.0582100000000000	0.0140200000000000	-0.0565800000000000	-0.6841900000000000	-3.2445900000000000
S&P 500	-0.0204600000000000	0.0845400000000000	-0.0335100000000000	-0.0424000000000000	-0.0105900000000000	-0.0123400000000000

Table B.8: Section 5.4 weighted slope model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.5172800000000000	0.5544200000000000	0.4973500000000000	0.5475400000000000	0.4606500000000000	0.5231100000000000
As Reported Earnings	0.8231100000000000	0.9342100000000000	0.1106700000000000	1.0000000000000000	0.1112300000000000	0.8852600000000000
Dividends	0.4831500000000000	0.4829200000000000	0.4901900000000000	0.4821500000000000	0.4810200000000000	0.4645100000000000
Buybacks	0.4905800000000000	0.6909600000000000	0.4986500000000000	0.3267400000000000	0.5735100000000000	0.6019500000000000
Retained Quarterly Earnings	0.6917900000000000	1.0000000000000000	0.0548700000000000	1.0000000000000000	0.9987300000000000	0.9999100000000000
S&P 500	0.4188900000000000	0.4580600000000000	0.4654300000000000	0.5621000000000000	0.4771500000000000	0.3562000000000000

Table B.9: Section 5.4 weighted slope model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0179900000000000
As Reported Earnings	-0.0069700000000000
Dividends	-0.0131000000000000
Buybacks	-0.0003300000000000
Retained Quarterly Earnings	0.5991100000000000
S&P 500	0.0167500000000000

Table B.10: Section 5.4.1 unweighted slope model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.0046800000000000	0.0114900000000000	-0.0195200000000000	-0.0040900000000000	-0.0013600000000000	-0.0066800000000000
As Reported Earnings	0.0211900000000000	-0.0275200000000000	-0.0488300000000000	-0.1533800000000000	-0.0627200000000000	-0.2272900000000000
Dividends	0.1188400000000000	0.0229900000000000	-0.1929000000000000	0.0683200000000000	-0.0005300000000000	-0.0294500000000000
Buybacks	0.0298000000000000	-0.0304800000000000	0.1529300000000000	-0.0371100000000000	0.0365400000000000	-0.0331600000000000
Retained Quarterly Earnings	-1.9379200000000000	-0.0272200000000000	0.3467900000000000	-0.9040800000000000	-0.5936500000000000	-1.5583400000000000
S&P 500	0.0009800000000000	0.0857200000000000	-0.0609600000000000	-0.0383300000000000	-0.0114300000000000	-0.1078600000000000

Table B.11: Section 5.4.1 unweighted slope model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.5020900000000000	0.4976900000000000	0.5024700000000000	0.5001600000000000	0.5005800000000000	0.5013600000000000
As Reported Earnings	0.6152200000000000	0.5109700000000000	0.4917300000000000	0.5365500000000000	0.5205900000000000	0.5395500000000000
Dividends	0.4249800000000000	0.5139000000000000	0.3840200000000000	0.5521700000000000	0.5002200000000000	0.5241400000000000
Buybacks	0.4520800000000000	0.4957400000000000	0.5392000000000000	0.4928100000000000	0.5119800000000000	0.5324700000000000
Retained Quarterly Earnings	0.4487900000000000	1.0000000000000000	0.2436100000000000	1.0000000000000000	1.0000000000000000	1.0000000000000000
S&P 500	0.4873500000000000	0.4998600000000000	0.5508300000000000	0.5112500000000000	0.5521500000000000	0.4678300000000000

Table B.12: Section 5.4.1 unweighted slope model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0101900000000000
As Reported Earnings	0.0000000000000000
Dividends	-0.0006300000000000
Buybacks	0.0000000000000000
Retained Quarterly Earnings	0.1199400000000000
S&P 500	0.2101700000000000

Table B.13: Section 5.5 weighted secants model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	-0.0072400000000000	0.0203100000000000	0.0015900000000000	-0.0168800000000000	-0.0019900000000000	-0.0054300000000000
As Reported Earnings	0.0334300000000000	0.0006800000000000	-0.0614200000000000	-0.0853800000000000	-0.0151700000000000	-0.5677000000000000
Dividends	0.0149000000000000	0.0409500000000000	-0.0681300000000000	0.0027700000000000	-0.0015800000000000	0.0037300000000000
Buybacks	0.0495500000000000	0.0391600000000000	-0.0256400000000000	-0.0162000000000000	0.0212500000000000	-0.0603400000000000
Retained Quarterly Earnings	-0.4900800000000000	-0.0927600000000000	-0.1353000000000000	-0.3104000000000000	-0.4118900000000000	-2.8079600000000000
S&P 500	-0.0210400000000000	0.0861700000000000	-0.0343500000000000	-0.0425800000000000	-0.0100200000000000	-0.0128200000000000

Table B.14: Section 5.5 weighted secants model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.5163200000000000	0.5427000000000000	0.4995900000000000	0.5398700000000000	0.4815400000000000	0.5229600000000000
As Reported Earnings	0.6593000000000000	0.9541000000000000	0.1304700000000000	1.0000000000000000	0.2183400000000000	1.0000000000000000
Dividends	0.4965100000000000	0.5008800000000000	0.4951500000000000	0.4740400000000000	0.4592200000000000	0.5291400000000000
Buybacks	0.4549400000000000	0.5354700000000000	0.4244200000000000	0.5110400000000000	0.5559300000000000	0.5407600000000000
Retained Quarterly Earnings	0.2759000000000000	1.0000000000000000	0.1467800000000000	1.0000000000000000	1.0000000000000000	1.0000000000000000
S&P 500	0.4430100000000000	0.4449500000000000	0.4507600000000000	0.5379800000000000	0.5218900000000000	0.3752100000000000

Table B.15: Section 5.5 weighted secants model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0180400000000000
As Reported Earnings	-0.0061200000000000
Dividends	-0.0131800000000000
Buybacks	-0.0003300000000000
Retained Quarterly Earnings	1.1656400000000000
S&P 500	0.0162300000000000

Table B.16: Section 5.5.1 unweighted secants model *A*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.0047400000000000	0.0043500000000000	-0.0046900000000000	-0.0016900000000000	0.0016600000000000	-0.0011500000000000
As Reported Earnings	-0.0346800000000000	-0.0267700000000000	-0.0610100000000000	-0.0555700000000000	-0.0206600000000000	-0.4077500000000000
Dividends	0.0207100000000000	0.0162300000000000	-0.0526700000000000	0.0131300000000000	0.0035800000000000	-0.0011400000000000
Buybacks	0.0388600000000000	0.0235000000000000	0.0002500000000000	-0.0140900000000000	0.0251100000000000	-0.0580900000000000
Retained Quarterly Earnings	-0.2504200000000000	-0.2567000000000000	-0.0921300000000000	-0.3373100000000000	-0.1352200000000000	-1.0998800000000000
S&P 500	0.0085400000000000	0.0253300000000000	-0.0100700000000000	0.0004300000000000	0.0090500000000000	-0.0442900000000000

Table B.17: Section 5.5.1 unweighted secants model *B*-coefficients in matrix form, full accuracy

	GDP Per Capita	As Reported Earnings	Dividends	Buybacks	Retained Quarterly Earnings	S&P 500
GDP Per Capita	0.5167600000000000	0.4817900000000000	0.5088700000000000	0.5037400000000000	0.4722700000000000	0.5038300000000000
As Reported Earnings	0.6218000000000000	0.6437300000000000	0.2055800000000000	1.0000000000000000	0.4768800000000000	0.9242400000000000
Dividends	0.3092400000000000	0.4102500000000000	0.3263500000000000	0.2801200000000000	0.5724700000000000	0.4951600000000000
Buybacks	0.4744200000000000	0.4648600000000000	0.5002100000000000	0.5373600000000000	0.5668500000000000	0.5658000000000000
Retained Quarterly Earnings	0.3670600000000000	0.6675700000000000	0.4160700000000000	0.9003100000000000	0.5428700000000000	0.7236900000000000
S&P 500	0.5202500000000000	0.3717700000000000	0.4976000000000000	0.4792500000000000	0.5015000000000000	0.4786600000000000

Table B.18: Section 5.5.1 unweighted secants model *D*-coefficients in matrix form, full accuracy

	Federal Funds Rate
GDP Per Capita	0.0032800000000000
As Reported Earnings	0.0000000000000000
Dividends	-0.0069500000000000
Buybacks	-0.0000900000000000
Retained Quarterly Earnings	0.4989200000000000
S&P 500	0.0783400000000000

Appendix C. Quad Chart

Appendix C includes the Quad Chart highlighting this research.

Modeling the Components Of An Economy As A Complex Adaptive System

Capt Richard J. Mickelsen
Advisor: Darryl K. Ahner, PhD, P.E.
Reader: Richard F. Deckro, DBA
Department of Operational Sciences (ENS)
Air Force Institute of Technology

Problem Statement

- Simplifying assumptions of current economic theory disregard interrelated structure and feedback between components
- Conventional financial modeling relies on autoregressive integrate moving average (ARIMA) techniques to make forecasts. These depend on historical trends and correlations to make future predictions
- Complex systems science is used widely in natural sciences, few studies have applied it to economic behavior

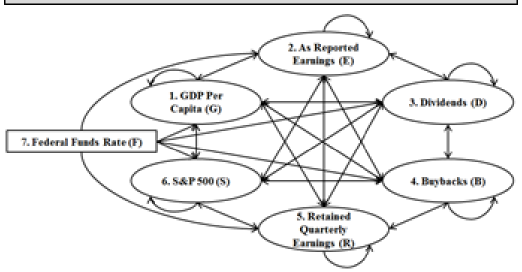
Question

Can elements of the economy be modeled as a complex adaptive system?

Objective

Model stock market as a fully-connected, dynamic network using a system of differential equations and an exogenous influencing factor

1. Conjecture Model



Findings

- Fitting to the data, rather than the slopes or secants, results in the most accurate portrayal of model behavior
- Euler curves can be used to accurately predict system performance over long time horizons
- Utility of model to predict system performance forward and study system components retroactively was established

2. Collect & Process Data

Sources

- U.S. Bureau of Economic Analysis
- U.S. Bureau of Labor Statistics
- Standard & Poor's
- St. Louis Federal Reserve

Format

- Quarterly data to capture long-term trends in market
- 17 years of data (69 quarters)
- Inflation-adjusted

3. Create Functional Form

Change in System State of S&P 500 = Effect of GDP in Current Period + Effect of As Reported Earnings in Current Period + Effect of Dividends in Current Period + Effect of Buybacks in Current Period + Effect of Retained Quarterly Earnings in Current Period + Effect of S&P 500 in Current Period + Impact of Changes in Federal Funds Rate

Endogenous Functions | Exogenous Influencing Factor

$$\dot{S}_t = f(G,E,D,B,R,S) + g(F)$$

Change in current state $= a_{ij}\left(\frac{x_t}{b_{ij}} - 80\right)$

$$\dot{S}_t = a_{11}\left(\frac{G_t}{b_{11}}-80\right)+a_{12}\left(\frac{E_t}{b_{12}}-80\right)+a_{13}\left(\frac{D_t}{b_{13}}-80\right)+a_{14}\left(\frac{B_t}{b_{14}}-80\right)+a_{15}\left(\frac{R_t}{b_{15}}-80\right)+a_{16}\left(\frac{S_t}{b_{16}}-80\right)+d_1F$$

$$\dot{G}_t = a_{11}\left(\frac{G_t}{b_{11}}-80\right)+a_{12}\left(\frac{E_t}{b_{12}}-80\right)+a_{13}\left(\frac{D_t}{b_{13}}-80\right)+a_{14}\left(\frac{B_t}{b_{14}}-80\right)+a_{15}\left(\frac{R_t}{b_{15}}-80\right)+a_{16}\left(\frac{S_t}{b_{16}}-80\right)+d_1F_t$$

$$\dot{E}_t = a_{21}\left(\frac{G_t}{b_{21}}-80\right)+a_{22}\left(\frac{E_t}{b_{22}}-80\right)+a_{23}\left(\frac{D_t}{b_{23}}-80\right)+a_{24}\left(\frac{B_t}{b_{24}}-80\right)+a_{25}\left(\frac{R_t}{b_{25}}-80\right)+a_{26}\left(\frac{S_t}{b_{26}}-80\right)+d_2F_t$$

$$\dot{D}_t = a_{31}\left(\frac{G_t}{b_{31}}-80\right)+a_{32}\left(\frac{E_t}{b_{32}}-80\right)+a_{33}\left(\frac{D_t}{b_{33}}-80\right)+a_{34}\left(\frac{B_t}{b_{34}}-80\right)+a_{35}\left(\frac{R_t}{b_{35}}-80\right)+a_{36}\left(\frac{S_t}{b_{36}}-80\right)+d_3F_t$$

$$\dot{B}_t = a_{41}\left(\frac{G_t}{b_{41}}-80\right)+a_{42}\left(\frac{E_t}{b_{42}}-80\right)+a_{43}\left(\frac{D_t}{b_{43}}-80\right)+a_{44}\left(\frac{B_t}{b_{44}}-80\right)+a_{45}\left(\frac{R_t}{b_{45}}-80\right)+a_{46}\left(\frac{S_t}{b_{46}}-80\right)+d_4F_t$$

$$\dot{R}_t = a_{51}\left(\frac{G_t}{b_{51}}-80\right)+a_{52}\left(\frac{E_t}{b_{52}}-80\right)+a_{53}\left(\frac{D_t}{b_{53}}-80\right)+a_{54}\left(\frac{B_t}{b_{54}}-80\right)+a_{55}\left(\frac{R_t}{b_{55}}-80\right)+a_{56}\left(\frac{S_t}{b_{56}}-80\right)+d_5F_t$$

$$\dot{S}_t = a_{61}\left(\frac{G_t}{b_{61}}-80\right)+a_{62}\left(\frac{E_t}{b_{62}}-80\right)+a_{63}\left(\frac{D_t}{b_{63}}-80\right)+a_{64}\left(\frac{B_t}{b_{64}}-80\right)+a_{65}\left(\frac{R_t}{b_{65}}-80\right)+a_{66}\left(\frac{S_t}{b_{66}}-80\right)+d_6F_t$$

Euler's Forward Method:

$$G_{t+1} = G_t + h\dot{G}_t$$
$$E_{t+1} = E_t + h\dot{E}_t$$
$$D_{t+1} = D_t + h\dot{D}_t$$
$$B_{t+1} = B_t + h\dot{B}_t$$
$$R_{t+1} = R_t + h\dot{R}_t$$
$$S_{t+1} = S_t + h\dot{S}_t$$

4. Solve for Model Coefficients

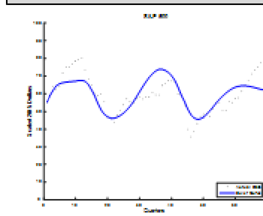
Minimize $f(x) = \sum_{i=1}^{n}\sum_{t=1}^{T}\frac{t}{T}\left(y_t^{(i)} - \hat{y}_t^{(i)}\right)^2$

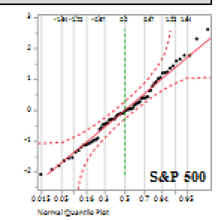
subject to

$b_{ij} \le 1$	for $i,j = 1,2,\dots,6$
$b_{ij} > 0$	for $i,j = 1,2,\dots,6$
$d_{ij} \le 0$	for $i = 2,3,4$
$a_{ij}, b_{ij}, d_i \in R$	for $i,j = 1,2,\dots,6$
$t \in Z^{+}$	

Coefficients are determined and model is fit via 3 approaches: fit to the data, slopes, and secants

5. Analyze Results

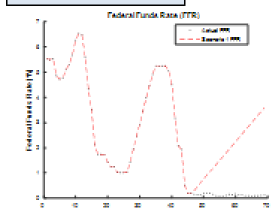


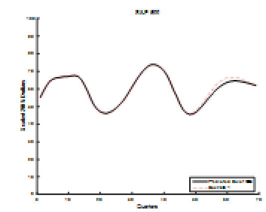


Euler's Forward Method projects system dynamics forward, using a single input and its own past predicted values to accurately forecast 17 years of stock market performance. Model satisfies mean-field theory, as reflected by normally-distributed residuals

Hypothesis testing fails to reject conclusion that Euler curves are statistically identical to actual data for five of six model components

	P-Value	Reject at $\alpha = 0.05$?
GDP Per Capita	0.2961	No
As Reported Earnings	0.5335	No
Dividends	0.7664	No
Buybacks	0.0392	Yes
Retained Quarterly Earnings	0.0755	No
S&P 500	0.9758	No





Model facilitates study of hypothetical "what if" scenarios, illustrating what would have happened had the Federal Reserve raised interest rates more quickly following the Great Recession. Prediction intervals evaluate alternative system end states compared baseline Euler curves

	95% Prediction Interval	Baseline	Overlap with Baseline?
GDP Per Capita ($)	[61590.93, 66517.87]	69255.58	No
As Reported Earnings (Bil $)	[101.49, 204.83]	240.56	No
Dividends (Bil $)	[71.73, 84.24]	90.42	No
Buybacks (Bil $)	[102.29, 125.92]	98.00	No
Retained Quarterly Earnings (Bil $)	[-105.88, -55.56]	-132.40	No
S&P 500 Index (Price Level, $)	[1512.15, 1620.46]	1579.14	Yes

Contributions

- Model is a proof of concept that financial markets can be modeled as complex adaptive systems
- A system of differential equations was used to simulate market behavior and predict turning points in the system
- Model demonstrates how financial markets can be forecasted over long time horizons with relative accuracy

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14. ABSTRACT
Complex systems science is relatively new and has not been widely applied to the field of economics. Much of current economic theory relies on principles of constrained optimization and fails to see economic variables as part of an interconnected network. While tools for forecasting economic indicators are based primarily on autoregressive techniques, these techniques may not be well-suited to predicting the future performance of highly volatile data sets such as the stock market. This research portrays the stock market as one component of a networked system of economic variables, with the federal funds rate acting as an exogenous influencing factor. Together these components form a complex adaptive system having nonlinear dynamics. The network is modeled using a system of differential equations, which are based on an expanded form of the logistic differential equation for populations with carrying capacities. An inverse problem is solved using the method of least squares, and the resulting coefficients are examined to determine the strength of relationships between the network components. The fitted model is then evaluated for adequacy and Euler’s Forward Method is employed to predict the long-run behavior of the network. With this as a baseline, the research investigates several hypothetical scenarios to determine how the system reacts to changes in interest rates. Contributions and implications of the model are addressed in the context of U.S. national defense.

15. SUBJECT TERMS
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